ONE PARAMETER FAMILIES AND NETS OF RULED SURFACES AND A NEW THEORY OF CONGRUENCES*

BY

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Introduction

The analytic basis for any projective theory of congruences, in which the lines of the congruence are defined either by a pair of points or a pair of planes, consists of the invariant theory of a completely integrable system of four homogeneous linear partial differential equations with two dependent and two independent variables, two of the equations of the system being of the first order, and two of the second order. If the developables of the congruence are known, this system of differential equations can be written in a very simple form.† But the determination of the developables of a congruence requires the integration of two partial differential equations of the first order, and it seems highly desirable to possess a theory which will be immediately applicable to any congruence, whether its developables can actually be found explicitly or not. The considerations made by G. M. Green‡ show that the existing theory can actually be modified so as to cover all such cases. However there are various ways in which this can be done and the various methods which might be used are not all equally desirable. Green himself has indicated one such method for the theory of congruences.§ But Green only indicated in a general way what was to be done without actually working out a complete theory. The great disadvantage which this particular theory would have, as compared with the one to which we are devoting this paper, is that only the final results would be of interest. The various types of invariants corresponding to the transformations of certain subgroups would

* Presented to the Society, December 30, 1919.

† E. J. Wilczynski, *Sur la théorie générale des congruences*. Mémoire conronnée par la classe des sciences. Mémoires publiées par la classe des Sciences de l'Académie Royale de Belgique. Collection en 4°. Deuxième séries. Tome III (1911). This paper will hereafter be cited as the Brussels Paper.

‡ G. M. Green, *On the theory of curved surfaces and canonical systems in projective differential geometry*, these Transactions, vol. 16 (1915).

§ G. M. Green, *Projective differential geometry of one-parameter families of space curves and conjugate nets on a curved surface*, American Journal of Mathematics, vol. 37 (1915), and vol. 38 (1916).

have no geometric significance. There exists another method, worked out by J. M. Kinney* in an as yet unpublished thesis, which labors under the same disadvantage.

The point of view of the present paper is to think of a congruence as being generated by a one-parameter family of ruled surfaces. If v is the parameter which varies from one of these surfaces to another, we consider first the seminvariants and invariants of the individual ruled surfaces v = const. These can be written down at once from the known theory of ruled surfaces. Let u be the variable which picks out a particular generator of such a ruled surface v = const. If we now allow v to be variable while u remains constant we obtain a second one-parameter family of ruled surfaces whose generators belong to the same congruence of lines. The ruled surfaces u=const. and v=const. together constitute a *net* of ruled surfaces, and we shall show next how to find the invariants of this net. In terms of these we shall then obtain the invariants of the one-parameter family of ruled surfaces v = const. They are those invariants of the net which do not change when the second family of the net is changed arbitrarily. Finally the invariants of the congruence are those invariants of the one-parameter family of ruled surfaces v = const., which are not changed when we replace this family by any other one-parameter family of the congruence.

1. The differential equations of the problem

Let $y^{(k)}$ and $z^{(k)}$, $(k = 1, 2, 3, 4)$, be the homogeneous coördinates of two distinct points, P_y and P_z, of space. Let $y^{(k)}$ and $z^{(k)}$ be given as analytic functions of two independent variables

$$y^{(k)} = f^{(k)}(u, v), \qquad z^{(k)} = g^{(k)}(u, v) \qquad (k = 1, 2, 3, 4), \tag{1}$$

and let us regard as *corresponding* points P_y and P_z, those which correspond to the same pair of values u, v. Unless the ratios of the $y^{(k)}$'s and also the ratios of the $z^{(k)}$'s reduce to functions of the same function t of u and v, the system of lines l obtained by joining all pairs of corresponding points will form a congruence. If in (1) we put v = const. we obtain a ruled surface of this congruence. Let us assume that the ruled surfaces v = const. are not developable; then the determinant

$$D = |y_u^{(k)}, z_u^{(k)}, y^{(k)}, z^{(k)}| \tag{2}$$

will not vanish identically,† and we can find a unique system of differential equations of the form

* J. M. Kinney, *On the general theory of congruences without preliminary integration.*

† It is not necessary to assume that *none* of the ruled surfaces v = const. are developable.

$$
(3) \qquad \begin{aligned} y_{uu} + p_{11}\, y_u + p_{12}\, z_u + q_{11}\, y + q_{12}\, z &= 0, \\ z_{uu} + p_{21}\, y_u + p_{22}\, z_u + q_{21}\, y + q_{22}\, z &= 0, \end{aligned}
$$

satisfied by the four pairs of functions $(y^{(k)},\ z^{(k)})$. The coefficients p_{ik} and q_{ik} can easily be expressed in terms of the functions $f^{(k)}$ and $g^{(k)}$, since we are assuming that D is not equal to zero. Of course p_{ik} and q_{ik} will be functions of v as well as of u. The projective properties of any individual ruled surface v = const. can be completely expressed in terms of the invariants of (3).* But the same thing is not true of the properties of the whole one-parameter family of surfaces. In fact equations (3) are satisfied also by the pairs of functions

$$
\eta^{(k)} = \sum_{i=1}^{4} V_{ki}(v)\, y^{(i)}, \qquad \zeta^{(k)} = \sum_{i=1}^{4} V_{ki}(v)\, z^{(i)} \quad (k = 1, 2, 3, 4),
$$

where $V_{ki}(v)$ are arbitrary functions of v. For each constant value of v these equations represent a projective transformation of the corresponding ruled surface v = const.; but this transformation is in general different for different ones of these surfaces, and does not represent a projective transformation of the one-parameter family.

Still starting from (1) and the assumption $D \neq 0$, we see that we can find coefficients a_{ik} and b_{ik}, so that the four pairs of functions $y^{(k)}$, $z^{(k)}$ will satisfy the following equations

$$
(4) \qquad \begin{aligned} y_v &= a_{11}\, y_u + a_{12}\, z_u + b_{11}\, y + b_{12}\, z, \\ z_v &= a_{21}\, y_u + a_{22}\, z_u + b_{21}\, y + b_{22}\, z, \end{aligned}
$$

as well as (3). It is easy to see that the most general pair of analytic functions which satisfies both of these systems of equations is

$$
(5) \qquad y = \sum_{k=1}^{4} c_k\, y^{(k)}, \qquad z = \sum_{k=1}^{4} c_k\, z^{(k)},
$$

where $c_1, \cdots, c_4$ are arbitrary constants. Consequently the system of four equations, composed of (3) and (4), may serve as basis for a projective theory of the one-parameter family of ruled surfaces v = const.

We have shown how to find such a system of partial differential equations for any one-parameter family of non-developable ruled surfaces, and it is evident that the resulting system will be completely integrable. It is also evident that any completely integrable system of this sort will define, except

* E. J. Wilczynski, *Projective differential geometry of curves and rules surfaces*, Leipzig, 1906, p. 133. This book will hereafter be quoted as Proj. Diff. Geom.

for projective transformations, a one-parameter family of non-developable ruled surfaces. The modifications which become necessary for the case of a one-parameter family of developables will become apparent later. We formulate our results as follows:

THEOREM. *Any analytic one-parameter family of non-developable ruled surfaces may be studied by means of a completely integrable system of partial differential equations of the form*

$$(S)\qquad \begin{aligned} y_{uu} + p_{11}\, y_u + p_{12}\, z_u + q_{11}\, y + q_{12}\, z &= 0, \\ z_{uu} + p_{21}\, y_u + p_{22}\, z_u + q_{21}\, y + q_{22}\, z &= 0, \\ y_v &= a_{11}\, y_u + a_{12}\, z_u + b_{11}\, y + b_{12}\, z, \\ z_v &= a_{21}\, y_u + a_{22}\, z_u + b_{21}\, y + b_{22}\, z; \end{aligned}$$

and every completely integrable system of this form defines a one-parameter family of non-developable ruled surfaces except for projective transformations. The notation is so chosen that the individual surfaces of the one-parameter family are obtained by equating v to a constant.

2. The integrability conditions of system (S)

If the coefficients p_{ik}, q_{ik}, a_{ik}, b_{ik}, of (S) are chosen as arbitrary functions of u and v, the system will not be a completely integrable one. We find it necessary to obtain the integrability conditions for this system. From the last two equations of (S) we find, by differentiation and making use of the first two equations, the following expressions:

$$(6)\qquad \begin{aligned} y_{uv} &= c_{11}\, y_u + c_{12}\, z_u + d_{11}\, y + d_{12}\, z, \\ z_{uv} &= c_{21}\, y_u + c_{22}\, z_u + d_{21}\, y + d_{22}\, z, \\ y_{vv} &= e_{11}\, y_u + e_{12}\, z_u + f_{11}\, y + f_{12}\, z, \\ z_{vv} &= e_{21}\, y_u + e_{22}\, z_u + f_{21}\, y + f_{22}\, z, \end{aligned}$$

where

$$(7)\qquad \begin{aligned} c_{11} &= (a_{11})_u - a_{11}\, p_{11} - a_{12}\, p_{21} + b_{11}, & d_{11} &= (b_{11})_u - a_{11}\, q_{11} - a_{12}\, q_{21}, \\ c_{12} &= (a_{12})_u - a_{11}\, p_{12} - a_{12}\, p_{22} + b_{12}, & d_{12} &= (b_{12})_u - a_{11}\, q_{12} - a_{12}\, q_{22}, \\ c_{21} &= (a_{21})_u - a_{21}\, p_{11} - a_{22}\, p_{21} + b_{21}, & d_{21} &= (b_{21})_u - a_{21}\, q_{11} - a_{22}\, q_{21}, \\ c_{22} &= (a_{22})_u - a_{21}\, p_{12} - a_{22}\, p_{22} + b_{22}, & d_{22} &= (b_{22})_u - a_{21}\, q_{12} - a_{22}\, q_{22}, \end{aligned}$$

and

$$
(8)\qquad
\begin{aligned}
e_{11} &= (a_{11})_v + a_{11}\,c_{11} + a_{12}\,c_{21} + b_{11}\,a_{11} + b_{12}\,a_{21}\,,\\
e_{12} &= (a_{12})_v + a_{11}\,c_{12} + a_{12}\,c_{22} + b_{11}\,a_{12} + b_{12}\,a_{22}\,,\\
e_{21} &= (a_{21})_v + a_{21}\,c_{11} + a_{22}\,c_{21} + b_{21}\,a_{11} + b_{22}\,a_{21}\,,\\
e_{22} &= (a_{22})_v + a_{21}\,c_{12} + a_{22}\,c_{22} + b_{21}\,a_{12} + b_{22}\,a_{22}\,,\\
f_{11} &= (b_{11})_v + a_{11}\,d_{11} + a_{12}\,d_{21} + b_{11}^2 + b_{12}\,b_{21}\,,\\
f_{12} &= (b_{12})_v + a_{11}\,d_{12} + a_{12}\,d_{22} + b_{11}\,b_{12} + b_{12}\,b_{22}\,,\\
f_{21} &= (b_{21})_v + a_{21}\,d_{11} + a_{22}\,d_{21} + b_{21}\,b_{11} + b_{22}\,b_{21}\,,\\
f_{22} &= (b_{22})_v + a_{21}\,d_{12} + a_{22}\,d_{22} + b_{21}\,b_{12} + b_{22}^2\,.
\end{aligned}
$$

These expressions for y_{uv}, z_{uv}, y_{vv}, z_{vv}, as well as the expressions for y_{uu} and z_{uu} obtained from (S) in terms of y_u, z_u, y, z, are determined uniquely. But if we proceed to calculate the third order derivatives of y and z, we see that some of these may be calculated in more than one way, and it becomes necessary to impose the condition that the two values obtained in this way shall be consistent. Of course the values of y_{uuu} and y_{vvv} can be obtained from (S) in one way only. The value of y_{uvv} may be obtained from either of the two equations

$$y_{uvv} = \frac{\partial y_{uv}}{\partial v}, \qquad y_{uvv} = \frac{\partial y_{vv}}{\partial u}.$$

But since y_{uv} and y_{vv}, as given by (6), were themselves obtained by differentiation of the same expression y_v contained in (S), we obtain no conditions when the two values of y_{uvv} are equated. The same remark applies to the two values of z_{uvv}. We do however obtain conditions upon the coefficients of (S) when we demand that

$$y_{uuv} = \frac{\partial y_{uu}}{\partial v} = \frac{\partial y_{uv}}{\partial u}, \qquad z_{uuv} = \frac{\partial z_{uu}}{\partial v} = \frac{\partial z_{uv}}{\partial u}.$$

The conditions obtained in this way are of the form

$$\alpha y_u + \beta z_u + \gamma y + \delta z = 0.$$

If they are to be satisfied identically, that is, for all solutions of system (S), the coefficients α, β, γ, δ, must all be equal to zero, since we are assuming that the determinant D, given by (2), is not equal to zero. Thus each of the two conditions splits up into four relations between the coefficients of (S). We obtain in this way the following eight equations; *the integrability conditions for the system* (S):

$$
(9)\quad
\begin{aligned}
&(p_{11})_v + p_{11}c_{11} + p_{12}c_{21} + q_{11}a_{11} + q_{12}a_{21} \\
&\qquad\qquad + (c_{11})_u - c_{11}p_{11} - c_{12}p_{21} + d_{11} = 0, \\
&(p_{12})_v + p_{11}c_{12} + p_{12}c_{22} + q_{11}a_{12} + q_{12}a_{22} \\
&\qquad\qquad + (c_{12})_u - c_{11}p_{12} - c_{12}p_{22} + d_{12} = 0, \\
&(p_{21})_v + p_{21}c_{11} + p_{22}c_{21} + q_{21}a_{11} + q_{22}a_{21} \\
&\qquad\qquad + (c_{21})_u - c_{21}p_{11} - c_{22}p_{21} + d_{21} = 0, \\
&(p_{22})_v + p_{21}c_{12} + p_{22}c_{22} + q_{21}a_{12} + q_{22}a_{22} \\
&\qquad\qquad + (c_{22})_u - c_{21}p_{12} - c_{22}p_{22} + d_{22} = 0,
\end{aligned}
$$

and

$$
(10)\quad
\begin{aligned}
&(q_{11})_v + p_{11}d_{11} + p_{12}d_{21} + q_{11}b_{11} + q_{12}b_{21} \\
&\qquad\qquad + (d_{11})_u - c_{11}q_{11} - c_{12}q_{21} = 0, \\
&(q_{12})_v + p_{11}d_{12} + p_{12}d_{22} + q_{11}b_{12} + q_{12}b_{22} \\
&\qquad\qquad + (d_{12})_u - c_{11}q_{12} - c_{12}q_{22} = 0, \\
&(q_{21})_v + p_{21}d_{11} + p_{22}d_{21} + q_{21}b_{11} + q_{22}b_{21} \\
&\qquad\qquad + (d_{21})_u - c_{21}q_{11} - c_{22}q_{21} = 0, \\
&(q_{22})_v + p_{21}d_{12} + p_{22}d_{22} + q_{21}b_{12} + q_{22}b_{22} \\
&\qquad\qquad + (d_{22})_u - c_{21}q_{12} - c_{22}q_{22} = 0.
\end{aligned}
$$

If these conditions are satisfied all of the partial derivatives of higher order, of either y or z, will be determined by unique expressions, linear and homogeneous in y, z, y_u, and z_u; the existence of analytic function solutions of (S) involving four arbitrary constants linearly and in homogeneous fashion follows at once, and the complete integrability of system (S) is established.

From the first and fourth of equations (9) we find by addition

$$
(11)\qquad (p_{11} + p_{22})_v + (c_{11} + c_{22} + b_{11} + b_{22})_u = 0.
$$

Consequently we may write

$$
(12)\qquad p_{11} + p_{22} = p_u, \qquad b_{11} + b_{22} + c_{11} + c_{22} = -p_v,
$$

where p is a function of u and v determined by (12) except for an additive constant.

3. The second one 'arameter family of ruled surfaces determined by (S)

We have seen that a system of form (S) may be utilized for the purpose of studying the one-parameter family of ruled surfaces $v = \text{const.}$ But clearly we obtain a second one-parameter family of ruled surfaces if we equate u to a constant. We may find the system (S') which is most convenient for the purpose of studying these surfaces, by simple eliminations from equations already deduced.

The first order equations of (S') may be obtained at once by solving (4) for y_u and z_u. If we substitute the values of y_u and z_u, obtained in this way, into the equations (6) for y_{vv} and z_{vv} we obtain the second order equations of system (S'). We find in this way;

$$(S')\qquad \begin{aligned} y_{vv} + r_{11}\,y_v + r_{12}\,z_v + s_{11}\,y + s_{12}\,z &= 0, \\ z_{vv} + r_{21}\,y_v + r_{22}\,z_v + s_{21}\,y + s_{22}\,z &= 0, \\ y_u = g_{11}\,y_v + g_{12}\,z_v + h_{11}\,y + h_{12}\,z&, \\ z_u = g_{21}\,y_v + g_{22}\,z_v + h_{21}\,y + h_{22}\,z&, \end{aligned}$$

where we are assuming that

$$a_2 = a_{11}\,a_{22} - a_{12}\,a_{21} \tag{13}$$

is different from zero, and where we have put

$$\begin{gathered} a_2\,g_{11} = a_{22}, \qquad a_2\,g_{12} = -\,a_{12}, \qquad a_2\,g_{21} = -\,a_{21}, \qquad a_2\,g_{22} = a_{11}, \\ a_2\,h_{11} = a_{12}\,b_{21} - a_{22}\,b_{11}, \qquad a_2\,h_{12} = a_{12}\,b_{22} - a_{22}\,b_{12}, \\ a_2\,h_{21} = a_{21}\,b_{11} - a_{11}\,b_{21}, \qquad a_2\,h_{22} = a_{21}\,b_{12} - a_{11}\,b_{22}, \end{gathered} \tag{14}$$

and

$$\begin{gathered} a_2\,r_{11} = -\,(e_{11}\,a_{22} - e_{12}\,a_{21}), \qquad a_2\,r_{12} = -\,(a_{11}\,e_{12} - a_{12}\,e_{11}), \\ a_2\,r_{21} = -\,(e_{21}\,a_{22} - e_{22}\,a_{21}), \qquad a_2\,r_{22} = -\,(a_{11}\,e_{22} - a_{12}\,e_{21}), \\ a_2\,s_{11} = -\begin{vmatrix} f_{11} & e_{11} & e_{12} \\ b_{11} & a_{11} & a_{12} \\ b_{21} & a_{21} & a_{22} \end{vmatrix}, \qquad a_2\,s_{12} = -\begin{vmatrix} f_{12} & e_{11} & e_{12} \\ b_{12} & a_{11} & a_{12} \\ b_{22} & a_{21} & a_{22} \end{vmatrix}, \\ a_2\,s_{21} = -\begin{vmatrix} f_{21} & e_{21} & e_{22} \\ b_{11} & a_{11} & a_{12} \\ b_{21} & a_{21} & a_{22} \end{vmatrix}, \qquad a_2\,s_{22} = -\begin{vmatrix} f_{22} & e_{21} & e_{22} \\ b_{12} & a_{11} & a_{12} \\ b_{22} & a_{21} & a_{22} \end{vmatrix}. \end{gathered} \tag{15}$$

If $a_2 = 0$, the two first order equations of (S') are replaced by

$$a_{22}\,y_v - a_{12}\,z_v + (a_{12}\,b_{21} - a_{22}\,b_{11})\,y + (a_{12}\,b_{22} - a_{22}\,b_{12})\,z = 0, \tag{16}$$

which shows that, in this case, the ruled surfaces $u =$ const. are developable. This gives us the theorem: *the ruled surfaces $u =$ const. are developables if and only if*

$$a_2 = a_{11}\,a_{22} - a_{12}\,a_{21} = 0.$$

The system of ruled surfaces, composed of the two one-parameter families u = const. and v = const., shall be called a *net of ruled surfaces.*

4. The differential equations of the surfaces of reference

If we return to equations (1), we observe that P_y and P_z describe, in general, two surfaces S_y and S_z when u and v vary over their ranges. In fact we may regard these surfaces and a general point-to-point correspondence between them as being given in advance, for the purpose of defining our two one-parameter families of ruled surfaces, or the net composed of both of them. We shall speak of the surfaces S_y and S_z as the *surfaces of reference.*

The differential equations of the surface S_y are obtained from

$$
\begin{aligned}
a_{11} y_u - y_v + b_{11} y &= - a_{12} z_u - b_{12} z, \\
y_{uu} + p_{11} y_u + * + q_{11} y &= - p_{12} z_u - q_{12} z, \\
- y_{uv} + c_{11} y_u + * + d_{11} y &= - c_{12} z_u - d_{12} z, \\
- y_{vv} + e_{11} y_u + * + f_{11} y &= - e_{12} z_u - f_{12} z,
\end{aligned}
\tag{17}
$$

by elimination of z_u and z. The differential equations of S_z are obtained from

$$
\begin{aligned}
a_{22} z_u - z_v + b_{22} z &= - a_{21} y_u - b_{21} y, \\
z_{uu} + p_{22} z_u + * + q_{22} z &= - p_{21} y_u - q_{21} y, \\
- z_{uv} + c_{22} z_u + * + d_{22} z &= - c_{21} y_u - d_{21} y, \\
- z_{vv} + e_{22} z_u + * + f_{22} z &= - e_{21} y_u - f_{21} y,
\end{aligned}
\tag{18}
$$

by elimination of y_u and y.

From some very familiar theorems we deduce the following results.

The curves v = const. are asymptotic curves on S_y, if and only if

$$
\begin{vmatrix} a_{12} & b_{12} \\ p_{12} & q_{12} \end{vmatrix} = 0. \tag{19a}
$$

They are asymptotic curves on S_z, if and only if

$$
\begin{vmatrix} a_{21} & b_{21} \\ p_{21} & q_{21} \end{vmatrix} = 0. \tag{19b}
$$

The curves u = const. are asymptotic curves on S_y or S_z respectively according as the conditions

$$
(20a)\quad \begin{vmatrix} a_{12} & b_{12} \\ e_{12} & f_{12} \end{vmatrix} = 0 \qquad \text{or} \qquad (20b)\quad \begin{vmatrix} a_{21} & b_{21} \\ e_{21} & f_{21} \end{vmatrix} = 0
$$

are satisfied.

The curves u = const. and v = const. form a conjugate system on S_y, if and only if

$$\begin{vmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{vmatrix} = 0; \tag{21a}$$

the corresponding condition for S_z *is*

$$\begin{vmatrix} a_{21} & b_{21} \\ c_{21} & d_{21} \end{vmatrix} = 0. \tag{21b}$$

5. The seminvariants

Let us transform system (S) by a transformation of the form

$$y = \alpha \bar{y} + \beta \bar{z}, \qquad z = \gamma \bar{y} + \delta \bar{z}, \tag{22}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary functions of u and v, for which

$$\Delta = \alpha\delta - \beta\gamma \neq 0. \tag{23}$$

The result will be a new system of form (S) whose coefficients, $\bar{p}_{ik}$, q_{ik}, $\bar{a}_{ik}$, and $\bar{b}_{ik}$ depend upon $\alpha, \beta, \gamma, \delta$ and upon the values of p_{ik}, q_{ik}, a_{ik}, and b_{ik}. Geometrically, this transformation has the effect of replacing the surfaces of reference, S_y and S_z, by two other surfaces, S^- and $S_{\bar{z}}$. The functions of p_{ik}, q_{ik}, a_{ik}, and b_{ik} which are left unchanged by this transformation, the *seminvariants*, are quantities whose values are independent of the choice of these surfaces of reference.

The effect of transformations of the form (22) upon the coefficients of (S) is given by the following equations which are important for what follows. We find

$$\begin{aligned}
\Delta\bar{p}_{11} &= 2(\alpha_u \delta - \gamma_u \beta) + p_{11}\alpha\delta + p_{12}\gamma\delta - p_{21}\alpha\beta - p_{22}\beta\gamma, \\
\Delta\bar{p}_{12} &= 2(\beta_u \delta - \delta_u \beta) + p_{11}\beta\delta + p_{12}\delta^2 - p_{21}\beta^2 - p_{22}\beta\delta, \\
\Delta\bar{p}_{21} &= -2(\alpha_u \gamma - \gamma_u \alpha) - p_{11}\alpha\gamma - p_{12}\gamma^2 + p_{21}\alpha^2 + p_{22}\alpha\gamma, \\
\Delta\bar{p}_{22} &= -2(\beta_u \gamma - \delta_u \alpha) - p_{11}\beta\gamma - p_{12}\gamma\delta + p_{21}\alpha\beta + p_{22}\alpha\delta,
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
\Delta\bar{q}_{11} &= \alpha_{uu}\delta - \gamma_{uu}\beta + p_{11}\alpha_u\delta + p_{12}\gamma_u\delta - p_{21}\alpha_u\beta - p_{22}\gamma_u\beta \\
&\qquad + q_{11}\alpha\delta + q_{12}\gamma\delta - q_{21}\alpha\beta - q_{22}\beta\gamma, \\
\Delta\bar{q}_{12} &= \beta_{uu}\delta - \delta_{uu}\beta + p_{11}\beta_u\delta + p_{12}\delta_u\delta - p_{21}\beta_u\beta - p_{22}\delta_u\beta \\
&\qquad + q_{11}\beta\delta + q_{12}\delta^2 - q_{21}\beta^2 - q_{22}\beta\delta, \\
\Delta\bar{q}_{21} &= -(\alpha_{uu}\gamma - \gamma_{uu}\alpha) - p_{11}\alpha_u\gamma - p_{12}\gamma_u\gamma + p_{21}\alpha_u\alpha + p_{22}\gamma_u\alpha \\
&\qquad - q_{11}\alpha\gamma - q_{12}\gamma^2 + q_{21}\alpha^2 + q_{22}\alpha\gamma, \\
\Delta\bar{q}_{22} &= -(\beta_{uu}\gamma - \delta_{uu}\alpha) - p_{11}\beta_u\gamma - p_{12}\delta_u\gamma + p_{21}\beta_u\alpha + p_{22}\delta_u\alpha \\
&\qquad - q_{11}\beta\gamma - q_{12}\gamma\delta + q_{21}\alpha\beta + q_{22}\alpha\delta.
\end{aligned} \tag{25}$$

either by direct calculation or else from the theory of ruled surfaces.* We find further

$$
(26)\qquad
\begin{aligned}
\Delta\bar{a}_{11} &= \alpha\delta a_{11} + \gamma\delta a_{12} - \alpha\beta a_{21} - \beta\gamma a_{22},\\
\Delta\bar{a}_{12} &= \beta\delta a_{11} + \delta^2 a_{12} - \beta^2 a_{21} - \beta\delta a_{22},\\
\Delta\bar{a}_{21} &= -\alpha\gamma a_{11} - \gamma^2 a_{12} + \alpha^2 a_{21} + \alpha\gamma a_{22},\\
\Delta\bar{a}_{22} &= -\beta\gamma a_{11} - \gamma\delta a_{12} + \alpha\beta a_{21} + \alpha\delta a_{22},
\end{aligned}
$$

and

$$
(27)\qquad
\begin{aligned}
\Delta\bar{b}_{11} &= -\alpha_v\delta + \gamma_v\beta + \alpha_u\delta a_{11} + \gamma_u\delta a_{12} - \alpha_u\beta a_{21} - \gamma_u\beta a_{22}\\
&\qquad + \alpha\delta b_{11} + \gamma\delta b_{12} - \alpha\beta b_{21} - \beta\gamma b_{22},\\
\Delta\bar{b}_{12} &= -\beta_v\delta + \delta_v\beta + \beta_u\delta a_{11} + \delta_u\delta a_{12} - \beta_u\beta a_{21} - \delta_u\beta a_{22}\\
&\qquad + \beta\delta b_{11} + \delta^2 b_{12} - \beta^2 b_{21} - \beta\delta b_{22},\\
\Delta\bar{b}_{21} &= \alpha_v\gamma - \gamma_v\alpha - \alpha_u\gamma a_{11} - \gamma_u\gamma a_{12} + \alpha_u\alpha a_{21} + \gamma_u\alpha a_{22}\\
&\qquad - \alpha\gamma b_{11} - \gamma^2 b_{12} + \alpha^2 b_{21} + \alpha\gamma b_{22},\\
\Delta\bar{b}_{22} &= \beta_v\gamma - \delta_v\alpha - \beta_u\gamma a_{11} - \delta_u\gamma a_{12} + \beta_u\alpha a_{21} + \delta_u\alpha a_{22}\\
&\qquad - \beta\gamma b_{11} - \delta\gamma b_{12} + \alpha\beta b_{21} + \alpha\delta b_{22}.
\end{aligned}
$$

From the theory of ruled surfaces we know that the quantities

$$
(28)\qquad
\begin{aligned}
u_{11} &= 2(p_{11})_u - 4q_{11} + p_{11}^2 + p_{12}p_{21},\\
u_{12} &= 2(p_{12})_u - 4q_{12} + p_{12}(p_{11} + p_{22}),\\
u_{21} &= 2(p_{21})_u - 4q_{21} + p_{21}(p_{11} + p_{22}),\\
u_{22} &= 2(p_{22})_u - 4q_{22} + p_{22}^2 + p_{12}p_{21},
\end{aligned}
$$

are transformed in accordance with the equations obtained from (26) if we replace a_{ik} by u_{ik}, and $\bar{a}_{ik}$ by $\bar{u}_{ik}$.† We express this fact by saying that *the a_{ik}'s and u_{ik}'s are cogredient* for transformations of the form (22).

From the u_{ik}'s and p_{ik}'s we form a new set of four quantities

$$
(29)\qquad
\begin{aligned}
u_{11}^{(1)} &= v_{11} = 2(u_{11})_u + p_{12}u_{21} - p_{21}u_{12},\\
u_{12}^{(1)} &= v_{12} = 2(u_{12})_u + (p_{11} - p_{22})u_{12} - p_{12}(u_{11} - u_{22}),\\
u_{21}^{(1)} &= v_{21} = 2(u_{21})_u - (p_{11} - p_{22})u_{21} + p_{21}(u_{11} - u_{22}),\\
u_{22}^{(1)} &= v_{22} = 2(u_{22})_u - p_{12}u_{21} + p_{21}u_{12},
\end{aligned}
$$

which are also cogredient with the u_{ik}'s and therefore with the a_{ik}'s.‡ The

* *Proj. Diff. Geom.*, pages 102–103.

† Ibid., pages 96 and 103.

‡ Ibid., pp. 99 and 100.

notation $u_{ik}^{(1)}$ *indicates a process,* defined by equations (29), by means of which from a set of four quantities, u_{ik}, may be derived four new quantities, $u_{ik}^{(1)}$, cogredient with the former set. Clearly this process may be repeated, giving rise to four new quantities

$$u_{ik}^{(11)} = v_{ik}^{(1)} = w_{ik},^{*} \tag{30}$$

also cogredient with the u_{ik}'s. Any further application of this process to the u_{ik}'s is unnecessary, since the new variables obtained in this way are linear combinations of the u_{ik}'s, v_{ik}'s, and w_{ik}'s with seminvariant coefficients.†

The variables u_{ik}, v_{ik}, and w_{ik}, are fundamental in the theory of ruled surfaces, but the variables a_{ik} do not occur in that theory at all. Since the a_{ik}'s are cogredient with the u_{ik}'s we can now obtain further sets of cogredient variables by applying the process defined by (29) to the a_{ik}'s. Thus we obtain the new sets of cogredient quantities $a_{ik}^{(1)}$, $a_{ik}^{(11)}$, etc.

From the theory of ruled surfaces we know that

$$\begin{gathered} I = u_{11} + u_{22}, \qquad J = u_{11}u_{22} - u_{12}u_{21}, \\ K = v_{11}v_{22} - v_{12}v_{21}, \qquad L = w_{11}w_{22} - w_{12}w_{21}, \\ \Delta = \begin{vmatrix} u_{11} - u_{22}, & v_{11} - v_{22}, & w_{11} - w_{22} \\ u_{12} & v_{12} & w_{12} \\ u_{21} & v_{21} & w_{21} \end{vmatrix} \end{gathered} \tag{31}$$

are seminvariants,‡ a fact which may moreover be verified directly from equations (29) and the cogredience properties noted. Of course $v_{11} + v_{22}$ and $w_{11} + w_{22}$ are also seminvariants; we have not listed them since the relations

$$v_{11} + v_{22} = 2I_u, \qquad w_{11} + w_{22} = 4I_{uu}$$

enable us to express them as derivatives of I. Moreover Δ is really not independent of I, J, K, L and of their derivatives, but it is so important a combination as to merit special mention. All of the seminvariants of a single ruled surface are functions of I, J, K, L and of their derivatives with respect to u.§

From the cogredience properties which we have noted, it follows at once that the quantities

$$\begin{gathered} a_1 = a_{11} + a_{22}, \qquad a_2 = a_{11}a_{22} - a_{12}a_{21}, \\ 2(a_1)_u = a_{11}^{(1)} + a_{22}^{(1)}, \qquad a_2^{(1)} = a_{11}^{(1)}a_{22}^{(1)} - a_{12}^{(1)}a_{21}^{(1)}, \qquad \text{etc.} \end{gathered} \tag{32}$$

are also seminvariants of (S). Moreover, if λ denotes an arbitrary constant,

* *Proj. Diff. Geom.*, p. 101.

† Ibid., p. 101.

‡ Ibid., pp. 97 to 102

§ Ibid., p. 101.

$$\begin{vmatrix} a_{11} + \lambda u_{11}, & a_{12} + \lambda u_{12} \\ a_{21} + \lambda u_{21}, & a_{22} + \lambda u_{22} \end{vmatrix}$$

will be a seminvariant for all values of λ. Consequently we find new seminvariants of the following type:

$$(33) \qquad \begin{aligned} (a, u) &= a_{11} u_{22} + a_{22} u_{11} - a_{12} u_{21} - a_{21} u_{12}, \\ (a, v) &= a_{11} v_{22} + a_{22} v_{11} - a_{12} v_{21} - a_{21} v_{12} = (a, u^{(1)}), \qquad \text{etc.} \end{aligned}$$

Of course, many of these are expressible in other forms. Thus we have, for instance,

$$(33a) \qquad (u, v) = (u, u^{(1)}) = 2J_u, \qquad (a, a^{(1)}) = 2(a_2)_u.$$

The seminvariant Δ is a very important representative of a type of seminvariants expressible as third order determinants formed from three distinct cogredient sets, and may be represented by the symbol

$$\Delta = (u, v, w).$$

Clearly, we can form other seminvariants of this form, such as

$$(34) \qquad (a, u, v), \qquad (a^{(1)}, u, v), \qquad (a, a^{(1)}, u), \qquad \text{etc.}.$$

All of the seminvariants obtained so far contain only the coefficients p_{ik}, q_{ik}, and a_{ik}, of system (S), besides partial derivatives of these quantities with respect to u.

In order to find seminvariants which involve the quantities b_{ik} also, and partial derivatives with respect to v, we might operate in exactly similar fashion upon the coefficients r_{ik}, s_{ik}, g_{ik}, h_{ik} of the system (S') of Art. 3. But we shall show how to obtain simpler seminvariants of the desired kind by a different method.

For this purpose we first re-write system (S) in a different form. We put

$$(35) \qquad \rho = 2y_u + p_{11} y + p_{12} z, \qquad \sigma = 2z_u + p_{21} y + p_{22} z.$$

The points, P_ρ and P_σ, whose homogeneous coördinates are given by these expressions are such that the line $P_\rho P_\sigma$ is a generator of the same set as $P_y P_z$ on the quadric surface H which osculates the ruled surface $v =$ const. along $P_y P_z$, while $P_y P_\rho$ and $P_z P_\sigma$ are two generators of the second set of H.*

As a result of (35), the equations of (S) assume the form

$$(36) \qquad \begin{aligned} 2\rho_u + p_{11} \rho + p_{12} \sigma &= u_{11} y + u_{12} z, \\ 2\sigma_u + p_{21} \rho + p_{22} \sigma &= u_{21} y + u_{22} z, \dagger \\ 2y_v + \pi_{11} y + \pi_{12} z &= a_{11} \rho + a_{12} \sigma, \\ 2z_v + \pi_{21} y + \pi_{22} z &= a_{21} \rho + a_{22} \sigma, \end{aligned}$$

* *Proj. Diff. Geom.*, p. 147.

† Ibid., p. 148.

where

$$(37)\qquad \begin{aligned} \pi_{11} &= a_{11}\,p_{11} + a_{12}\,p_{21} - 2b_{11}, & \pi_{12} &= a_{11}\,p_{12} + a_{12}\,p_{22} - 2b_{12}, \\ \pi_{21} &= a_{21}\,p_{11} + a_{22}\,p_{21} - 2b_{21}, & \pi_{22} &= a_{21}\,p_{12} + a_{22}\,p_{22} - 2b_{22}. \end{aligned}$$

But corresponding to all transformations of the form (22), the variables, ρ and σ, defined by (35) undergo the cogredient transformations

$$\rho = \alpha\bar{\rho} + \beta\bar{\sigma}, \qquad \sigma = \gamma\bar{\rho} + \delta\bar{\sigma}.*$$

Consequently it follows, by comparing the first two equations of (36) with the last two, and making use of the various cogredience properties already noted; that the quantities π_{ik}, introduced by (37), must be cogredient with the quantities p_{ik} except that the partial derivatives of $\alpha, \beta, \gamma, \delta$ with respect to u which occur in (24) must be replaced by v-derivatives.

From this remark we conclude further that, from a set of quantities like a_{ik}, or any cogredient set, we can obtain a new set of cogredient variables if, in the process which is exemplified by (29), we replace the quantities p_{ik} by π_{ik} and at the same time replace the u-derivatives by v-derivatives. We use an upper index 2 to indicate this new process. Thus, for instance, we obtain

$$(38)\qquad \begin{aligned} a_{11}^{(2)} &= 2\,(a_{11})_v + \pi_{12}\,a_{21} - \pi_{21}\,a_{12}, \\ a_{12}^{(2)} &= 2\,(a_{12})_v + (\pi_{11} - \pi_{22})\,a_{12} - \pi_{12}\,(a_{11} - a_{22}), \qquad \text{etc.} \end{aligned}$$

From the quantities obtained in this way we form seminvariants as before. Thus

$$(39)\qquad \begin{aligned} a_{11}^{(2)} + a_{22}^{(2)} &= 2\,(a_{11} + a_{22})_v, & a_{11}^{(2)}\,a_{22}^{(2)} - a_{12}^{(2)}\,a_{21}^{(2)} &= a_2^{(2)}, \\ a_{11}^{(22)}\,a_{22}^{(22)} - a_{12}^{(22)}\,a_{22}^{(22)} &= a_2^{(22)}, & u_{11}^{(2)}\,u_{22}^{(2)} - u_{12}^{(2)}\,u_{21}^{(2)} &= u_2^{(2)}, \qquad \text{etc.} \end{aligned}$$

are seminvariants. Moreover the two processes may be combined. Thus the quantities $(a_{ik}^{(1)})^{(2)}$ are cogredient with the quantities a_{ik}, and give rise to a seminvariant

$$(40)\qquad (a_{11}^{(1)})^{(2)}\,(a_{22}^{(1)})^{(2)} - (a_{12}^{(1)})^{(2)}\,(a_{21}^{(1)})^{(2)} = a_2^{(12)};$$

the similar formation which results when the two processes are used in opposite order may be denoted by $a_2^{(21)}$.

We have noticed already that, from two or three sets of quantities cogredient with the a_{ik}'s, bilinear or trilinear seminvariants of type (33) or (34) may be formed. By applying this remark to the new variables just obtained we find such new seminvariants as

$$\begin{aligned} &(a^{(2)}, u), \qquad (a, u^{(2)}), \qquad \text{etc.,} \\ &(a, a^{(1)}, a^{(2)}), \qquad (a^{(1)}, u, v), \qquad (a^{(2)}, u, v), \qquad \text{etc.} \end{aligned}$$

* *Proj. Diff. Geom.*, p. 146.

6. The flecnodal canonical form

Let us assume that

$$\theta_4 = (u_{11} - u_{22})^2 + 4u_{12}u_{21}, \tag{41}$$

is not equal to zero. Then the flecnode curve on each of the ruled surfaces v = const. has two distinct branches. The locus of these curves is in general a surface of two distinct sheets which we shall call the *locus of the flecnodes* of our one-parameter family of ruled surfaces. We use this locution to avoid confusion with the *flecnode surface,* which is a ruled surface the locus of the flecnode tangents of a single ruled surface. In general, the totality of the flecnode tangents of a one-parameter family of ruled surfaces will be a congruence of lines; the totality of the flecnode surfaces of a one-parameter family of ruled surfaces will be a new one-parameter family of ruled surfaces. We shall speak of them as the *congruence of flecnode tangents,* and the *one-parameter family of flecnode surfaces* respectively. Again the congruence of flecnode tangents should not be confused with what we have formerly called the flecnode congruence of a single ruled surface. The locus of these for a one-parameter family of ruled surfaces gives rise, in general to a complex, the *flecnode complex* of the family.*

If the two sheets of the locus of flecnodes are distinct, that is, if $\theta_4 \neq 0$, we may use them as surfaces of reference for the system (S). We shall then have

$$u_{12} = u_{21} = 0, \qquad u_{11} - u_{22} \neq 0 .\dagger \tag{42}$$

If these conditions are satisfied, they will still be satisfied after any transformation of the form

$$y = \alpha\bar{y}, \qquad z = \delta\bar{z},$$

and α and δ may moreover be chosen, according to (24), in such a way as to make $\bar{p}_{11} = \bar{p}_{22} = 0$. Thus, if $\theta_4 \neq 0$, we may suppose

$$u_{12} = u_{21} = 0, \qquad u_{11} - u_{22} \neq 0, \qquad p_{11} = p_{22} = 0. \tag{43}$$

When the coefficients of (S) satisfy these conditions and one other condition to be formulated presently we shall say that (S) is in the *flecnodal canonical form.*

We propose to show that *all of the coefficients of the flecnodal canonical form are expressible in terms of seminvariants of the original system* (S).

In fact we have,‡ under the assumption $\theta_4 \neq 0$, $\theta_{10} \neq 0$,

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* *Proj. Diff. Geom.*, pp. 146–153, and pp. 175–190.

† Ibid., pp. 149–150.

‡ Ibid., p. 120, equ. (102).

$$p_{11} = 0, \qquad p_{22} = 0, \qquad q_{12} = \tfrac{1}{2}(p_{12})_u, \qquad q_{21} = \tfrac{1}{2}(p_{21})_u,$$

$$p_{12} = \frac{\sqrt{\theta_{10}}}{\theta_4} e^{\frac{1}{4}\int \sqrt{\theta_4}\frac{\theta_9}{\theta_{10}} du}, \qquad p_{21} = \frac{\sqrt{\theta_{10}}}{\theta_4} e^{-\frac{1}{4}\int \sqrt{\theta_4}\frac{\theta_9}{\theta_{10}} du},$$

$$(44) \qquad 64 q_{11} = \frac{1}{\theta_4^2}[16\theta_{10} - \vartheta_{10} + 8\theta_4 \theta_4'' - 9(\theta_4')^2] - 8\sqrt{\theta_4},$$

$$64 q_{22} = \frac{1}{\theta_4^2}[16\theta_{10} - \vartheta_{10} + 8\theta_4 \theta_4'' - 9(\theta_4')^2] + 8\sqrt{\theta_4},$$

where θ_4 has already been defined, where θ_9 is the same as the quantity Δ defined by (31), and where

$$(45) \qquad \begin{aligned} \theta_{10} &= (I^2 - 4J)(K - I_u)^2 + (II_u - 2J_u)^2 \\ &= \tfrac{1}{4}\{(u_{12} v_{21} - u_{21} v_{12})^2 - 4[(u_{11} - u_{22}) v_{12} \\ &\qquad - (v_{11} - v_{22}) u_{12}][(u_{11} - u_{22}) v_{21} - (v_{11} - v_{22}) u_{21}]\}, \\ \vartheta_{10} &= 8\theta_4(\theta_4)_{uu} - 9(\theta_4)_u^2 + 8I\theta_4^2. \end{aligned}$$

These quantities $\theta_4, \theta_9, \theta_{10}, \vartheta_{10}$ are, in fact, not merely seminvariants of (S); they are invariants of the one-parameter family of ruled surfaces $v = \text{const.}$; ϑ_{10} in particular is the so-called quadri-derivative of θ_4 and is usually denoted by $\theta_{4\cdot 1}$, a notation which we shall have to abandon to avoid confusion in our later developments. Since the eight coefficients, p_{ik} and q_{ik}, of the flecnodal canonical form are now expressed in terms of $\theta_4, \theta_9, \theta_{10}$, and ϑ_{10}, and since these quantities are functions of I, J, K, L, and of their partial derivatives with respect to u, it only remains to show that the remaining eight coefficients, a_{ik} and b_{ik}, of the flecnodal canonical form can also be expressed in terms of seminvariants. For this purpose we consider the seminvariants

$$(46) \qquad \begin{aligned} a_1 &= a_{11} + a_{22}, \qquad a_2 = a_{11} a_{22} - a_{12} a_{21}; \\ (a, u) &= a_{11} u_{22} + a_{22} u_{11} - a_{12} u_{21} - a_{21} u_{12}, \\ (a, u, v) &= \begin{vmatrix} a_{11} - a_{22}, & u_{11} - u_{22}, & v_{11} - v_{22} \\ a_{12} & u_{12} & v_{12} \\ a_{21} & u_{21} & v_{21} \end{vmatrix}. \end{aligned}$$

In the canonical form, we have $u_{12} = u_{21} = 0$, and owing to the assumptions $\theta_4 \neq 0$, $\theta_{10} \neq 0$, the quantities $u_{11} - u_{22}$, v_{12}, and v_{21} will not be zero. Consequently equations (46) enable us to express the four coefficients a_{ik} of the canonical form in terms of $a_1, a_2, (a, u), (a, u, v)$ and I, J, K, L and their derivatives.

Let us consider next the seminvariants

$$a_1^{(2)} = a_{11}^{(2)} + a_{22}^{(2)}, \qquad a_2^{(2)} = a_{11}^{(2)} a_{22}^{(2)} - a_{12}^{(2)} a_{21}^{(2)},$$

$$(a^{(2)}, u) = a_{11}^{(2)} u_{22} + a_{22}^{(2)} u_{11} - a_{12}^{(2)} u_{21} - a_{21}^{(2)} u_{12},$$

(47)
$$(a, a^{(1)}, a^{(2)}) = \begin{vmatrix} a_{11} - a_{22}, & a_{11}^{(1)} - a_{22}^{(1)}, & a_{11}^{(2)} - a_{22}^{(2)} \\ a_{12}, & a_{12}^{(1)}, & a_{12}^{(2)} \\ a_{21}, & a_{21}^{(1)}, & a_{21}^{(2)} \end{vmatrix},$$

the first of which, $a_1^{(2)}$, need not be mentioned explicitly, since it is equal to $2(a_1)_v$. If the cofactors of $a_{12}^{(2)}$ and $a_{21}^{(2)}$ in $(a, a^{(1)}, a^{(2)})$ are not both equal to zero, these equations enable us to express the values of $a_{ik}^{(2)}$, for the canonical form, in terms of the seminvariants mentioned previously and of $a_2^{(2)}$, $(a^{(2)}, u)$, and $(a, a^{(1)}, a^{(2)})$. According to (38), we obtain in this way seminvariant expressions for π_{12}, π_{21}, and $\pi_{11} - \pi_{22}$. The integrability condition (11) may be written

(48)
$$(\pi_{11} + \pi_{22})_u - (a_{11} + a_{22})_{uu} = (p_{11} + p_{22})_v .$$

In our canonical form, we have $p_{11} = p_{22} = 0$, so that (48) enables us to conclude that

$$\pi_{11} + \pi_{22} - (a_{11} + a_{22})_u = V(v)$$

is a function of v alone. But any transformation of the form

$$y = \alpha(v)\bar{y}, \qquad z = \delta(v)\bar{z},$$

where α and δ are functions of v alone, preserves the conditions $p_{11} = p_{22} = u_{12} = u_{21} = 0$, and transforms $\pi_{11} + \pi_{22} - (a_{11} + a_{22})_u$ into

$$\overline{\pi_{11} + \pi_{22} - (a_{11} + a_{22})_u} = \pi_{11} + \pi_{22} - (a_{11} + a_{22})_u + 2\left(\frac{\alpha_v}{\alpha} + \frac{\delta_v}{\delta}\right).$$

Consequently, by choosing $\alpha(v)$ and $\delta(v)$ in accordance with the condition

$$\frac{\alpha_v}{\alpha} + \frac{\delta_v}{\delta} = -\tfrac{1}{2}[\pi_{11} + \pi_{22} - (a_{11} + a_{22})_u] = -\tfrac{1}{2}V(v),$$

we may make $\bar{\pi}_{11} + \bar{\pi}_{22} - (\bar{a}_{11} + \bar{a}_{22})_u$ equal to zero. Let us assume that this transformation has been made, so that

(48a)
$$\pi_{11} + \pi_{22} = (a_{11} + a_{22})_u;$$

this is the condition mentioned above, which together with

$$u_{12} = u_{21} = p_{11} = p_{22} = 0$$

characterizes the flecnodal canonical form of system (S).

By means of (48a), $\pi_{11} + \pi_{22}$ is expressed in terms of seminvariants; since we have shown before that π_{12}, π_{21}, and $\pi_{11} - \pi_{22}$ may be expressed in terms of certain seminvariants, it follows that all of the π_{ik}'s can be thus expressed. Finally, equations (37) show that the same thing is true of the b_{ik}'s.

We have obtained the following result. *If θ_4 and θ_{10} are different from zero, and if the co-factors of $a_{12}^{(2)}$ and $a_{21}^{(2)}$ in $(a, a^{(1)}, a^{(2)})$ are not both zero, all of the coefficients of the flecnodal canonical form are functions of the eleven seminvariants*

$$I, J, K, L, a_1, a_2, (a, u), (a, u, v), a_2^{(2)}, (a^{(2)}, u), (a, a^{(1)}, a^{(2)}),$$

and of partial derivatives of these seminvariants.

From this theorem it follows at once that *any seminvariant of (S) can be expressed as a function of the eleven seminvariants mentioned and of their partial derivatives, provided that θ_4 and θ_{10} do not vanish, and that at least one of the cofactors of $a_{12}^{(2)}$ or $a_{21}^{(2)}$ in $(a, a^{(1)}, a^{(2)})$ is not equal to zero.*

It is known from the theory of ruled surfaces that the cases $\theta_4 = 0$ and $\theta_{10} = 0$ are really exceptional; that is, in these cases the coefficients of the canonical form can not be expressed entirely in terms of the seminvariants mentioned. The additional distinction, however, as to whether the co-factors of $a_{12}^{(2)}$ and $a_{21}^{(2)}$ in $(a, a^{(1)}, a^{(2)})$ do or do not vanish, might be avoided by substituting for $(a, a^{(1)}, a^{(2)})$ some other seminvariant, such as $(a^{(2)}, v)$ for instance. We prefer, however, to retain $(a, a^{(1)}, a^{(2)})$ for reasons which will become apparent later.

7. Invariants of the net

The seminvariants are those combinations formed from coefficients of (S) whose values are independent of the choice of the surfaces of reference, S_y and S_z. But the value of a seminvariant is not, in general, independent of the parametric representation of the ruled surfaces of the net. If we make a transformation of the form

$$\bar{u} = U(u), \qquad \bar{v} = V(v), \tag{49}$$

where $U(u)$ and $V(v)$ are arbitrary functions of the single variables indicated, the parametric ruled surfaces of the net are left unchanged, but their parametric representation is altered. Those functions of seminvariants which are not changed at all by any transformation of form (49) shall be called *absolute invariants of the net.* A seminvariant $\theta_{k,l}$, which is transformed in accordance with the equation

$$\bar{\theta}_{k,l} = (U')^{-k}(V')^{-l}\theta_{k,l}$$

shall be called a *relative net invariant of weights k and l.*

The theory of ruled surfaces supplies us with a number of such net-invariants. The fundamental invariants of this sort are those denoted by θ_4, θ_9, θ_{10}, and ϑ_{10}, in Article 6. Moreover the theory of ruled surfaces enables us to form all net-invariants which depend merely on the coefficients p_{ik} and q_{ik}. They are obtained from the four fundamental ones by certain simple differentiation processes.*

In accordance with the general notation $\theta_{k,\,l}$ just established, the four fundamental net-invariants θ_4, θ_9, θ_{10}, ϑ_{10} shall now be denoted by $\theta_{4,\,0}$, $\theta_{9,\,0}$, $\theta_{10,\,0}$, and $\vartheta_{10,\,0}$ respectively.

In order to be in a position to find the remaining net-invariants, we study the effect of a transformation of form (49) upon the coefficients of (S). We find the following transformation equations;

$$(50)\qquad \begin{gathered} \bar{p}_{11} = \frac{1}{U'}(p_{11} + \eta), \qquad \bar{p}_{12} = \frac{p_{12}}{U'}, \qquad \bar{p}_{21} = \frac{p_{21}}{U'}, \qquad \bar{p}_{22} = \frac{1}{U'}(p_{22} + \eta), \\ \bar{q}_{ik} = \frac{q_{ik}}{(U')^2}, \qquad \bar{a}_{ik} = \frac{U'}{V'}a_{ik}, \qquad \bar{b}_{ik} = \frac{b_{ik}}{V'}, \end{gathered}$$

where

$$(51)\qquad \eta = \frac{U''}{U'}.$$

From these equations, the following may be deduced;

$$(52)\qquad \begin{aligned} \bar{u}_{11} &= \frac{1}{(U')^2}(u_{11} + 2\mu), & \bar{u}_{12} &= \frac{1}{(U')^2}u_{12}, \\ \bar{u}_{21} &= \frac{1}{(U')^2}u_{21}, & \bar{u}_{22} &= \frac{1}{(U')^2}(u_{22} + 2\mu), \dagger \end{aligned}$$

where

$$(53)\qquad \mu = \eta' - \tfrac{1}{2}\eta^2 = \frac{U'''}{U'} - \frac{3}{2}\left(\frac{U''}{U'}\right)^2 = \{U, u\}$$

is the Schwarzian derivative of U with respect to u. We find further

$$(54)\qquad \bar{I} = \frac{1}{(U')^2}(I + 4\mu), \qquad \bar{J} = \frac{1}{(U')^4}(J + 2\mu I + 4\mu^2),$$

and

$$(55)\qquad \begin{aligned} \bar{v}_{11} &= \frac{1}{(U')^3}(v_{11} - 4u_{11}\eta + 4\mu' - 8\mu\eta), & \bar{v}_{12} &= \frac{1}{(U')^3}(v_{12} - 4u_{12}\eta), \\ {}_{21} &= \frac{1}{(U')^3}(v_{21} - 4u_{21}\eta), & \bar{v}_{22} &= \frac{1}{(U')^3}(v_{22} - 4u_{22}\eta + 4\mu' - 8\mu\eta), \end{aligned}$$

where μ' is the derivative of μ.

* *Proj. Diff. Geom.*, p. 121.

† Compare *Proj. Diff. Geom.*, p. 104.

The invariance of $\theta_{4,0}$, $\theta_{9,0}$, $\theta_{10,0}$, and $\vartheta_{10,0}$ has been established already. To obtain the remaining fundamental net-invariants, we must find the effect of these transformations upon the seven seminvariants a_1, a_2, (a, u), (a, u, v), $a_2^{(2)}$, $(a^{(2)}, u)$, and $(a, a^{(1)}, a^{(2)})$. From (46) and (50) we see that

$$(56)\qquad \bar{a}_1 = \frac{U'}{V'} a_1, \qquad \bar{a}_2 = \left(\frac{U'}{V'}\right)^2 a_2 .$$

Therefore a_1 and a_2 are relative net-invariants which may be denoted by

$$\theta_{-1,1} = a_1, \qquad \theta_{-2,2} = a_2 .$$

Of course

$$(57)\qquad A = a_1^2 - 4a_2 = (a_{11} - a_{22})^2 + 4a_{12}a_{21}$$

is also a relative net-invariant. The special importance of A will appear later.

We find further

$$(58)\qquad (\overline{a, u}) = \frac{1}{U'V'}[(a, u) + 2\mu a_1].$$

Combining this with (54) and (56), we see that

$$(59)\quad \theta_{1,1} = (a, u) - \tfrac{1}{2} I a_1 = - a_{12}u_{21} - a_{21}u_{12} - \tfrac{1}{2}(a_{11} - a_{22})(u_{11} - u_{22})$$

is a new net-invariant. The seminvariant (a, u, v) is transformed in accordance with the equation

$$(\overline{a, u, v}) = \frac{1}{(U')^4 V'}(a, u, v),$$

so that

$$(60)\qquad \theta_{4,1} = (a, u, v)$$

is a net-invariant.

We find further

$$\bar{\pi}_{ik} = \frac{1}{V'}(\pi_{ik} + \eta a_{ik}),$$

$$(61)\quad (\bar{a}_{ik})_{\bar{u}} = \frac{1}{V'}[(a_{ik})_u + \eta a_{ik}], \qquad (\bar{a}_{ik})_{\bar{v}} = \frac{U'}{(V')^2}[(a_{ik})_u - \zeta a_{ik}],$$

where

$$(62)\qquad \zeta = \frac{V''}{V'},$$

so that

$$(63)\qquad \bar{a}_{ik}^{(1)} = \frac{1}{V'}(a_{ik}^{(1)} + 2\eta a_{ik}), \qquad \bar{a}_{ik}^{(2)} = \frac{U'}{(V')^2}(a_{ik}^{(2)} - 2\zeta a_{ik}).$$

Consequently we find

$$\bar{a}_2^{(2)} = \frac{(U')^2}{(V')^4}[a_2^{(2)} - 4\zeta(a_2)_v + 4\zeta^2 a_2],$$

$$(64) \qquad \overline{(a^{(2)}, u)} = \frac{1}{(U')(V')^2}[(a^{(2)}, u) - 2\zeta(a, u) + 4\mu(a_1)_v - 4\mu\zeta a_1],$$

$$\overline{(a, a^{(1)}, a^{(2)})} = \frac{(U')^2}{(V')^4}(a, a^{(1)}, a^{(2)}).$$

Thus, the seminvariant

$$(65) \qquad \theta_{-2,\,4} = (a, a^{(1)}, a^{(2)})$$

is also a net-invariant, and it remains to find two further net-invariants, one involving $a_2^{(2)}$ and one involving $(a^{(2)}, u)$.

Let us introduce the seminvariant

$$(66) \qquad A^{(2)} = (a_1^{(2)})^2 - 4a_2^{(2)}$$

analogous to A. We find

$$(67) \qquad \bar{A}^{(2)} = \frac{(U')^2}{(V')^4}[A^{(2)} - 4\zeta A_v + 4\zeta^2 A].$$

Moreover we have

$$(68) \qquad (\bar{A})_{\bar{u}} = \frac{U'}{(V')^2}(A_u + 2\eta A), \qquad (\bar{A})_{\bar{v}} = \frac{(U')^2}{(V')^3}(A_v - 2\zeta A).$$

Consequently

$$(69) \qquad \theta_{-4,\,6} = AA^{(2)} - (A_v)^2 = (a_1^2 - 4a_2)[(a_1^{(2)})^2 - 4a_2^{(2)}] - 4[a_1(a_1)_v - 2(a_2)_v]^2$$

is a new net-invariant.

Finally we find, from (58),

$$(70) \qquad \overline{(a, u)}_{\bar{v}} = \frac{1}{U'(V')^2}[(a, u)_v - \zeta(a, u) + 2\mu(a_1)_v - 2\mu\zeta a_1],$$

which proves that

$$(71) \qquad \theta_{1,\,2} = (a^{(2)}, u) - 2(a, u)_v = -(a, u^{(2)})$$

is a net-invariant.

From the results of Article 6 it now follows that all of the coefficients of the flecnodal canonical form can be expressed in terms of the net-invariants

$$\theta_{0,\,4},\ \theta_{0,\,9},\ \theta_{0,\,10},\ \vartheta_{0,\,10},\ a_1,\ a_2,\ \theta_{1,\,1},\ \theta_{4,\,1},\ \theta_{-4,\,6},\ \theta_{1,\,2},\ \theta_{-2,\,4}.$$

These eleven net-invariants may therefore be regarded as fundamental. In fact, if they are given as functions of u and v, the net of ruled surfaces is determined except for projective transformations, and all other net-invariants are expressible in terms of these eleven and others derived from them by certain differentiation processes. Of course the eleven invariants cannot be assigned arbitrarily. The relations which exist between them can be found by writing the integrability conditions for a system (S) in its flecnodal canonical form.

8. Invariants of the one-parameter family of ruled surfaces v = constant

If we transform our system (S) by a transformation of the form

$$\bar{u} = \phi(u, v), \qquad \bar{v} = V(v), \tag{72}$$

where $\phi(u, v)$ is an arbitrary function of both u and v, while $V(v)$ is an arbitrary function of v alone, the one-parameter family of ruled surfaces $\bar{v}$ = const. will coincide with the family v = const., but the family $\bar{u}$ = const. will not coincide with u = const. In fact the ruled surfaces $\bar{u}$ = const. may, by choice of $\phi(u, v)$, be made to coincide with any one-parameter family made up of the generators of the ruled surfaces v = const. Consequently, those combinations of net-invariants which remain invariant under all transformations of form (72) will be invariants of the one-parameter family of ruled surfaces v = const.

If we again use the notation $\bar{p}_{ik}$, $\bar{q}_{ik}$, etc., to denote the coefficients of the system of differential equations obtained from (S) by a transformation of form (72), we find

$$\begin{aligned}
\bar{p}_{11} &= \frac{1}{\phi_u}\left(p_{11} + \frac{\phi_{uu}}{\phi_u}\right), & \bar{p}_{12} &= \frac{p_{12}}{\phi_u}, \\
\bar{p}_{21} &= \frac{p_{21}}{\phi_u}, & \bar{p}_{22} &= \frac{1}{\phi_u}\left(p_{22} + \frac{\phi_{uu}}{\phi_u}\right), \\
\bar{q}_{ik} &= \frac{q_{ik}}{\phi_u^2}, & \bar{b}_{ik} &= \frac{b_{ik}}{V'}, \\
\bar{a}_{11} &= \frac{1}{V'}(a_{11}\phi_u - \phi_v), & \bar{a}_{12} &= \frac{\phi_u}{V'} a_{12}, \\
\bar{a}_{21} &= \frac{\phi_u}{V'} a_{21}, & \bar{a}_{22} &= \frac{1}{V'}(a_{22}\phi_u - \phi_v),
\end{aligned} \tag{73}$$

whence

$$\bar{u}_{11} = \frac{1}{\phi_u^2}(u_{11} + 2\mu), \qquad \bar{u}_{12} = \frac{u_{12}}{\phi_u},$$

$$(74) \quad \bar{u}_{21} = \frac{u_{21}}{\phi_u^2}, \qquad \bar{u}_{22} = \frac{1}{\phi_u^2}(u_{22} + 2\mu),$$

$$\bar{a}_1 = \frac{1}{V'}(a_1\phi_u - 2\phi_v), \qquad \bar{a}_2 = \frac{1}{(V')^2}(a_2\phi_u^2 - a_1\phi_u\phi_v + \phi_v^2),$$

where

$$(75) \qquad \mu = \frac{\phi_{uuu}}{\phi_u} - \frac{3}{2}\left(\frac{\phi_{uu}}{\phi_u}\right)^2 = \{\phi, u\}.$$

We see that the net-invariants a_1 and a_2 are not invariants of the one-parameter family; but if we put, as before,

$$A = a_1^2 - 4a_2,$$

we find

$$(76) \qquad \overline{A} = \frac{\phi_u^2 A}{(V')^2},$$

so that A is such an invariant.

It is evident that $\theta_{4,0}$, $\theta_{9,0}$, $\theta_{10,0}$, and $\vartheta_{10,0}$ are invariants of the one-parameter family of ruled surfaces. The formulas (74) enable us to verify that the same thing is true of $\theta_{1,1}$. In order to investigate the effect of (72) on $\theta_{4,1}$, $\theta_{-4,6}$, $\theta_{1,2}$, and $\theta_{-2,4}$, we must first obtain formulas for $\bar{v}_{ik}$, $\bar{a}_{ik}^{(1)}$, and $\bar{a}_{ik}^{(2)}$.

If θ is any function of u and v, we have

$$(77) \qquad \frac{\partial\theta}{\partial\bar{u}} = \frac{1}{\phi_u}\frac{\partial\theta}{\partial u}, \qquad \frac{\partial\theta}{\partial\bar{v}} = \frac{1}{V'}\left[-\frac{\phi_v}{\phi_u}\frac{\partial\theta}{\partial u} + \frac{\partial\theta}{\partial v}\right].$$

Consequently we find

$$\bar{v}_{11} - \bar{v}_{22} = \frac{1}{\phi_u^3}\left[v_{11} - v_{22} - 4(u_{11} - u_{22})\frac{\phi_{uu}}{\phi_u}\right],$$

$$(78) \qquad \bar{v}_{12} = \frac{1}{\phi_u^3}\left(v_{12} - 4u_{12}\frac{\phi_{uu}}{\phi_u}\right), \qquad \bar{v}_{21} = \frac{1}{\phi_u^3}\left(v_{21} - 4u_{21}\frac{\phi_{uu}}{\phi_u}\right),$$

by making use of (73), (74), (77), and the equations which define v_{ik}. From (73) and (77) we find

$$(\bar{a}_{11})_{\bar{u}} = \frac{1}{V'}\left[(a_{11})_u + \frac{\phi_{uu}}{\phi_u}a_{11} - \frac{\phi_{uv}}{\phi_u}\right],$$

$$(\bar{a}_{12})_{\bar{u}} = \frac{1}{V'}\left[(a_{12})_u + \frac{\phi_{uu}}{\phi_u}a_{12}\right],$$

$$(\bar{a}_{21})_{\bar{u}} = \frac{1}{V'}\left[(a_{21})_u + \frac{\phi_{uu}}{\phi_u}a_{21}\right],$$

$$(\bar{a}_{22})_{\bar{u}} = \frac{1}{V'}\left[(a_{22})_u + \frac{\phi_{uu}}{\phi_u}a_{22} - \frac{\phi_{uv}}{\phi_u}\right],$$

(79)
$$(V')^2(\bar{a}_{11})_{\bar{v}} = -\phi_v\left[(a_{11})_u + a_{11}\frac{\phi_{uu}}{\phi_u} - \frac{\phi_{uv}}{\phi_u}\right] + \phi_u\left[(a_{11})_v + a_{11}\frac{\phi_{uv}}{\phi_u} - \frac{\phi_{vv}}{\phi_u} - \zeta a_{11} + \zeta\frac{\phi_v}{\phi_u}\right],$$

$$(V')^2(\bar{a}_{12})_{\bar{v}} = -\phi_v\left[(a_{12})_u + a_{12}\frac{\phi_{uu}}{\phi_u}\right] + \phi_u\left[(a_{12})_v + a_{12}\frac{\phi_{uv}}{\phi_u} - \zeta a_{12}\right],$$

$$(V')^2(\bar{a}_{21})_{\bar{v}} = -\phi_v\left[(a_{21})_u + a_{21}\frac{\phi_{uu}}{\phi_u}\right] + \phi_u\left[(a_{21})_v + a_{21}\frac{\phi_{uv}}{\phi_u} - \zeta a_{21}\right],$$

$$(V')^2(\bar{a}_{22})_{\bar{v}} = -\phi_v\left[(a_{22})_u + a_{22}\frac{\phi_{uu}}{\phi_u} - \frac{\phi_{uv}}{\phi_u}\right] + \phi_u\left[(a_{22})_v + a_{22}\frac{\phi_{uv}}{\phi_u} - \frac{\phi_{vv}}{\phi_u} - \zeta a_{22} + \zeta\frac{\phi_v}{\phi_u}\right].$$

From the equations (73) and the definition of π_{ik}, we deduce the following:

$$V'\bar{\pi}_{11} = \pi_{11} - \frac{\phi_v}{\phi_u}p_{11} + \left(a_{11} - \frac{\phi_v}{\phi_u}\right)\frac{\phi_{uu}}{\phi_u},$$

(80)
$$V'\bar{\pi}_{12} = \pi_{12} - \frac{\phi_v}{\phi_u}p_{12} + a_{12}\frac{\phi_{uu}}{\phi_u}, \qquad V'\bar{\pi}_{21} = \pi_{21} - \frac{\phi_v}{\phi_u}p_{21} + a_{21}\frac{\phi_{uu}}{\phi_u},$$

$$V'\bar{\pi}_{22} = \pi_{22} - \frac{\phi_v}{\phi_u}p_{22} + \left(a_{22} - \frac{\phi_v}{\phi_u}\right)\frac{\phi_{uu}}{\phi_u}.$$

Making use of (79) and (80), we now find

(81)
$$V'\bar{a}_{11}^{(1)} = a_{11}^{(1)} + 2a_{11}\frac{\phi_{uu}}{\phi_u} - 2\frac{\phi_{uv}}{\phi_u}, \qquad V'\bar{a}_{12}^{(1)} = a_{12}^{(1)} + 2a_{12}\frac{\phi_{uu}}{\phi_u},$$

$$V'\bar{a}_{21}^{(1)} = a_{21}^{(1)} + 2a_{21}\frac{\phi_{uu}}{\phi_u}, \qquad V'\bar{a}_{22}^{(1)} = a_{22}^{(1)} + 2a_{22}\frac{\phi_{uu}}{\phi_u} - 2\frac{\phi_{uv}}{\phi_u},$$

and

$$(V')^2 \bar{a}_{11}^{(2)} = \phi_u a_{11}^{(2)} - \phi_v a_{11}^{(1)} + 2\left(\phi_{uv} - \phi_v \frac{\phi_{uu}}{\phi_u} - \phi_u \zeta\right) a_{11} + 2\phi_v\left(\frac{\phi_{uv}}{\phi_u} - \frac{\phi_{vv}}{\phi_v} + \zeta\right),$$

(82)

$$(V')^2 \bar{a}_{12}^{(2)} = \phi_u a_{12}^{(2)} - \phi_v a_{12}^{(1)} + 2\left(\phi_{uv} - \phi_v \frac{\phi_{uu}}{\phi_u} - \phi_u \zeta\right) a_{12},$$

$$(V')^2 \bar{a}_{21}^{(2)} = \phi_u a_{21}^{(2)} - \phi_v a_{21}^{(1)} + 2\left(\phi_{uv} - \phi_v \frac{\phi_{uu}}{\phi_u} - \phi_u \zeta\right) a_{21},$$

$$(V')^2 \bar{a}_{22}^{(2)} = \phi_u a_{22}^{(2)} - \phi_v a_{22}^{(1)} + 2\left(\phi_{uv} - \phi_v \frac{\phi_{uu}}{\phi_u} - \phi_u \zeta\right) a_{22} + 2\phi_v\left(\frac{\phi_{uv}}{\phi_u} - \frac{\phi_{vv}}{\phi_v} + \zeta\right).$$

These formulas enable us to verify, at a glance, that the net-invariants $\theta_{4,1}$ and $\theta_{-2,4}$ are invariants of the one-parameter family as well. It remains to determine the effect of the transformation on the net-invariants $\theta_{-4,6}$ and $\theta_{1,2}$, and to find two further invariants of the one-parameter family from these two net-invariants.

This may be accomplished as follows. Equations (74) show that we can make $\bar{a}_1 = 0$ by choosing for our new variable $\bar{u} = \phi(u, v)$ a solution (different from $\phi = \text{const.}$) of the differential equation

$$a_1 \phi_u - 2\phi_v = 0. \tag{83}$$

Moreover, the most general solution of (83) will be an arbitrary function of $\bar{u}$. Consequently, the one-parameter family of ruled surfaces $v = \text{const.}$ determines uniquely a second one-parameter family of ruled surfaces $\bar{u} = \text{const.}$, the two families constituting a net for which the net-invariant a_1 has the value zero. We shall call this second one-parameter family of ruled surfaces the *conjugate* family. Since it is determined uniquely by the original family, it is clear that the invariants of the *net* formed by the given one-parameter family and its conjugate family will be invariants of the one-parameter family as well. We shall call such a net *a conjugate net of ruled surfaces,* and we shall explain later the geometric relation between the ruled surfaces of the two conjugate families.

From every net-invariant we obtain an invariant of the one-parameter family, by subjecting it to the general transformation of the form (72), and then substituting in the expression thus found the following values, derived from (83);

$$(84)\qquad \phi_v = \tfrac{1}{2} a_1 \phi_u, \qquad \phi_{uv} = \tfrac{1}{2}[(a_1)_u \phi_u + a_1 \phi_{uu}],$$
$$\phi_{vv} = \tfrac{1}{2}\{(a_1)_v + \tfrac{1}{2} a_1 (a_1)_u\}\phi_u + \tfrac{1}{4} a_1^2 \phi_{uu},$$

whence

$$(85)\qquad \phi_{uv} - \phi_v \frac{\phi_{uu}}{\phi_u} - \phi_u \zeta = \tfrac{1}{2}\phi_u[(a_1)_u - 2\zeta],$$
$$\frac{\phi_{uv}}{\phi_u} - \frac{\phi_{vv}}{\phi_v} + \zeta = \zeta - \frac{(a_1)_v}{a_1}.$$

The resulting expression will be an invariant of the one-parameter family, except for a factor of the form $\phi_u^k (V')^l$.

Let us apply this process to the net-invariant $\theta_{-4,\,6}$. We have

$$\theta_{-4,\,6} = AA^{(2)} - (A_v)^2,$$

where

$$A^{(2)} = (a_1^{(2)})^2 - 4a_2^{(2)} = (a_{11}^{(2)} - a_{22}^{(2)})^2 + 4a_{12}^{(2)} a_{21}^{(2)}.$$

According to (82) and (85) we have

$$\frac{(V')^2}{\phi_u}(\bar a_{11}^{(2)} - \bar a_{22}^{(2)}) = a_{11}^{(2)} - a_{22}^{(2)} - \tfrac{1}{2} a_1 (a_{11}^{(1)} - a_{22}^{(1)}) + [(a_1)_u - 2\zeta](a_{11} - a_{22}),$$

etc.,

so that

$$(86)\qquad \frac{(V')^4}{\phi_u^2}\bar A^{(2)} = [a_{11}^{(2)} - a_{22}^{(2)} - \tfrac{1}{2} a_1 (a_{11}^{(1)} - a_{22}^{(1)}) + (a_1)_u (a_{11} - a_2)]^2$$
$$+ 4[a_{12}^{(2)} - \tfrac{1}{2} a_1 a_{12}^{(1)} + (a_1)_u a_{12}][a_{21}^{(2)} - \tfrac{1}{2} a_1 a_{21}^{(1)} + (a_1)_u a_{21}]$$
$$- 4\zeta[A_v - \tfrac{1}{2} a_1 A_u + (a_1)_u A] + 4\zeta^2 A,$$

since we have

$$(87)\qquad (a_{11} - a_{22})(a_{11}^{(1)} - a_{22}^{(1)}) + 2a_{12} a_{21}^{(1)} + 2a_{21} a_{12}^{(1)} = A_u,$$
$$(a_{11} - a_{22})(a_{11}^{(2)} - a_{22}^{(2)}) + 2a_{12} a_{21}^{(2)} + 2a_{21} a_{12}^{(2)} = A_v.$$

We also have

$$\bar A = \frac{\phi_u^2}{(V')^2} A,$$

whence

$$(88)\qquad \frac{(V')^3}{\phi_u^2}\bar A_{\bar v} = -\frac{\phi_v}{\phi_u} A_u + A_v + 2\left(\frac{\phi_{uv}}{\phi_u} - \frac{\phi_v}{\phi_u}\frac{\phi_{uu}}{\phi_u} - \zeta\right) A,$$

or, under the conditions (84) and (85),

$$(89)\qquad \frac{(V')^3}{\phi_u^2}\bar A_{\bar v} = A_v - \tfrac{1}{2} a_1 A_u + (a_1)_u A - 2\zeta A.$$

Consequently we find for

$$\frac{(V')^6}{\phi_u^2}\bar\theta_{-4,\,6}$$

the following expression which is the new invariant which we were seeking

$$\theta'_{-4,6} = A[\{a_{11}^{(2)} - a_{22}^{(2)} - \tfrac{1}{2}a_1(a_{11}^{(1)} - a_{22}^{(1)}) + (a_1)_u(a_{11} - a_{22})\}^2 + 4\{a_{12}^{(2)} - \tfrac{1}{2}a_1 a_{12}^{(1)} + (a_1)_u a_{12}\}\{a_{21}^{(2)} - \tfrac{1}{2}a_1 a_{21}^{(1)} + (a_1)_u a_{21}\}] - [A_v - \tfrac{1}{2}a_1 A_u + (a_1)_u A]^2,$$

and which, on account of (87) may be written more simply

(90) $$\theta'_{-4,6} = A[\{a_{11}^{(2)} - a_{22}^{(2)} - \tfrac{1}{2}a_1(a_{11}^{(1)} - a_{22}^{(1)})\}^2 + 4\{a_{12}^{(2)} - \tfrac{1}{2}a_1 a_{12}^{(1)}\}\{a_{21}^{(2)} - \tfrac{1}{2}a_1 a_{21}^{(1)}\}] - (A_v - \tfrac{1}{2}a_1 A_u)^2.$$

The formula for the net-invariant $\theta_{1,2}$ was

$$\theta_{1,2} = (a^{(2)}, u) - 2(a, u)_v.$$

Making use of (84) and (85), we find

$$V'\phi_u(\overline{a, u}) = (a, u) - \tfrac{1}{2}a_1 I = \theta_{1,1},$$

whence, according to (77),

(91) $$\phi_u(V')^2(\overline{a, u})_{\bar{v}} = (\theta_{1,1})_v - \tfrac{1}{2}a_1(\theta_{1,1})_u - \tfrac{1}{2}(a_1)_u\theta_{1,1} - \zeta\theta_{1,1}.$$

On the other hand, we find

(92) $$\phi_u(V')^2(\overline{a^{(2)}, u}) = (a^{(2)}, u) - \tfrac{1}{2}a_1(a^{(1)}, u) + (a_1)_u(a, u) - (a_1)_v I - 2\zeta\theta_{1,1},$$

giving rise to the new invariant

(93) $$\theta'_{1,2} = (a^{(2)}, u) - \tfrac{1}{2}a_1(a^{(1)}, u) - I[(a_1)_v - \tfrac{1}{2}a_1(a_1)_u] - 2[(\theta_{1,1})_v - \tfrac{1}{2}a_1(\theta_{1,1})_u] + 2(a_1)_u\theta_{1,1}.$$

The ten invariants

(94) $$\theta_{0,4},\ \theta_{0,9},\ \theta_{0,10},\ \vartheta_{0,10},\ A,\ \theta_{1,1},\ \theta_{4,1},\ \theta'_{-4,6},\ \theta'_{1,2},\ \theta_{-2,4}$$

clearly determine a one-parameter family of ruled surfaces, except for projective transformations. For, if we specify besides the value of a_1 as a function of u and v, for instance $a_1 = 0$, the corresponding net of ruled surfaces is determined except for a projective transformation.

9. Introduction of the focal surfaces as surfaces of reference. The focal canonical form

The generators of the ruled surfaces of a one-parameter family form a congruence of lines. The invariants of the congruence are those invariants of the one-parameter family which are left unaltered by any transformation of the form

$$(95) \qquad \bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v),$$

where both ϕ and ψ are arbitrary functions of u and v. These invariants might be obtained by extending the methods of Article 8 to the more general transformation (95). It seems preferable however to establish connection, at this point, with a previously developed projective theory of congruences.* We accomplish this by showing how to transform system (S) into the system of differential equations which was used in that other theory.

Let us assume that the quadratic equation

$$(96) \qquad -a_{21} t^2 + (a_{11} - a_{22}) t + a_{12} = 0,$$

has two distinct roots, so that

$$A = (a_{11} - a_{22})^2 + 4a_{12} a_{21}$$

is different from zero. If we make a transformation of form (22), in which the ratios $\alpha : \gamma$ and $\beta : \delta$ are equated to the two roots of (96), we shall have $\alpha\delta - \beta\gamma \neq 0$, and according to (26), the transformed system will have $\bar{a}_{12} = \bar{a}_{21} = 0$.

Although this transformation is not unique, the resulting new surfaces of reference, $S_{\bar{y}}$ and $S_{\bar{z}}$, are determined uniquely; they are, in fact, the focal surfaces of the congruence. This follows from the fact that the first order equations of the transformed system (S) will have the form

$$\bar{y}_v = \bar{a}_{11} \bar{y}_u + \bar{b}_{11} \bar{y} + \bar{b}_{12} \bar{z}, \qquad \bar{z}_v = \bar{a}_{22} \bar{z}_u + \bar{b}_{21} \bar{y} + \bar{b}_{22} \bar{z}.$$

For, if $\bar{b}_{12}$ and $\bar{b}_{21}$ are not zero, these equations show that every line $P_{\bar{y}} P_{\bar{z}}$ of the congruence is tangent to each of the surfaces $S_{\bar{y}}$ and $S_{\bar{z}}$. The cases $\bar{b}_{12} = 0$ or $\bar{b}_{21} = 0$ correspond to the cases when one or both focal surfaces degenerate into curves. Consequently $\bar{b}_{12}$ and $\bar{b}_{21}$ must be invariants of the congruence whose explicit expressions we shall obtain very soon.

We may formulate our result more elegantly as follows. *The function*

$$(97) \qquad a_{21} y^2 - (a_{11} - a_{22}) yz - a_{12} z^2$$

is a quadratic covariant of system (S). *The factors of this covariant determine in general two points on each line of the congruence, the foci of the line, that is, the points where the line touches the focal locus of the congruence. The foci are distinct or coincident according as* A *is not or is equal to zero.*

From this last remark it is clear that A is an invariant of the congruence.

Let us factor the covariant (97), and denote its factors by $\bar{y}$ and $\bar{z}$. Then $S_{\bar{y}}$ and $S_{\bar{z}}$ will be the two sheets of the focal locus. In order to avoid exceptional cases, and to preserve symmetry, we write the factors of (97) in the following form

* Brussels Paper.

(98) $$\bar{y} = \sqrt{a}y + \frac{a_{12}}{\sqrt{a}}z, \qquad \bar{z} = \frac{-a_{21}}{\sqrt{a}}y + \sqrt{a}z,$$

where

(99) $$a = \tfrac{1}{2}(a_{11} - a_{22} + \sqrt{A}).$$

In these formulas $\sqrt{A}$ stands for that one of the two square roots of

$$A = (a_{11} - a_{22})^2 + 4a_{12}a_{21}$$

which reduces to $a_{11} - a_{22}$ when a_{12} or a_{21} is equal to zero. The square root $\sqrt{a}$ may be chosen in either of the two available ways; but after a choice has been made, it must be adhered to. Since we are discussing the case $A \neq 0$, a will also be different from zero as a result of the specification just made of the meaning of the symbol $\sqrt{A}$.

From (98) we find

(100) $$\sqrt{A}y = \sqrt{a}\bar{y} - \frac{a_{12}}{\sqrt{a}}\bar{z}, \qquad \sqrt{A}z = \frac{a_{21}}{\sqrt{a}}\bar{y} + \sqrt{a}\bar{z}.$$

Consequently we shall find the coefficients A_{ik}, B_{ik}, P_{ik}, Q_{ik} of the resulting canonical form of the system (S) for $\bar{y}$ and $\bar{z}$, by substituting into the general formulas of Article 5 the values

$$\alpha = \sqrt{\frac{a}{A}} = \frac{a}{\sqrt{Aa}}, \qquad \beta = \frac{-a_{12}}{\sqrt{Aa}}, \qquad \gamma = \frac{a_{21}}{\sqrt{Aa}}, \qquad \delta = \frac{a}{\sqrt{Aa}}.$$

This canonical form shall be called the *focal* canonical form.

We find in this way

(101) $$\begin{aligned}
\sqrt{A}U_{11} &= \tfrac{1}{2}(a_{11} - a_{22})(u_{11} - u_{22}) \\
&\qquad + a_{21}u_{12} + a_{12}u_{21} + \tfrac{1}{2}\sqrt{A}(u_{11} + u_{22}), \\
\sqrt{A}U_{22} &= -\tfrac{1}{2}(a_{11} - a_{22})(u_{11} - u_{22}) \\
&\qquad - a_{21}u_{12} - a_{12}u_{21} + \tfrac{1}{2}\sqrt{A}(u_{11} + u_{22}), \\
\sqrt{A}U_{12} &= -a_{12}(u_{11} - u_{22}) + au_{12} - \frac{a_{12}^2}{a}u_{21}, \\
\sqrt{A}U_{21} &= -a_{21}(u_{11} - u_{22}) - \frac{a_{21}^2}{a}u_{12} + au_{21},
\end{aligned}$$

and

(102) $$A_{11} = \tfrac{1}{2}(a_{11} + a_{22} + \sqrt{A}), \qquad A_{22} = \tfrac{1}{2}(a_{11} + a_{22} - \sqrt{A}),$$
$$A_{12} = A_{21} = 0.$$

The transformations of form (22) transform the quantities a_{ik} and u_{ik} cogrediently. Comparison of (24) and (26) shows that the p_{ik}'s are transformed cogrediently with the u_{ik}'s except for the presence in each $\bar{p}_{ik}$ of some terms which involve partial derivatives of $\alpha, \beta, \gamma, \delta$ with respect to u. Therefore $\sqrt{A}P_{ik}$ will be obtained from $\sqrt{A}U_{ik}$ if we replace the u_{ik}'s which appear in the right members of (101) by the corresponding p_{ik}'s and then add the additional terms involving the derivatives of $\alpha, \beta, \gamma, \delta$.

We find the following formulas;

$$
(103) \quad
\begin{aligned}
\sqrt{A}P_{11} &= \frac{a_u}{a^2}(a^2 - a_{12}a_{21}) - \frac{A_u}{aA}(a^2 + a_{12}a_{21}) + \frac{2a_{12}(a_{21})_u}{a} \\
&\qquad + \tfrac{1}{2}(a_{11} - a_{22})(p_{11} - p_{22}) \\
&\qquad + a_{21}p_{12} + a_{12}p_{21} + \tfrac{1}{2}\sqrt{A}(p_{11} + p_{22}), \\
\sqrt{A}P_{22} &= \frac{a_u}{a^2}(a^2 - a_{12}a_{21}) - \frac{A_u}{aA}(a^2 + a_{12}a_{21}) + \frac{2a_{21}(a_{12})_u}{a} \\
&\qquad - \tfrac{1}{2}(a_{11} - a_{22})(p_{11} - p_{22}) \\
&\qquad - a_{21}p_{12} - a_{12}p_{21} + \tfrac{1}{2}\sqrt{A}(p_{11} + p_{22}), \\
\sqrt{A}P_{12} &= \frac{2}{a}[a_{12}a_u - a(a_{12})_u] - a_{12}(p_{11} - p_{22}) + ap_{12} - \frac{a_{12}^2}{a}p_{21}, \\
\sqrt{A}P_{21} &= -\frac{2}{a}[a_{21}a_u - a(a_{21})_u] - a_{21}(p_{11} - p_{22}) - \frac{a_{21}^2}{a}p_{12} + ap_{21}.
\end{aligned}
$$

We notice the relations

$$(104) \qquad a^2 - a_{12}a_{21} = a(a_{11} - a_{22}), \qquad a^2 + a_{12}a_{21} = \tfrac{1}{2}a\sqrt{A},$$

and find

$$(105) \qquad P_{11} + P_{22} = p_{11} + p_{22} - \frac{A_u}{A},$$

a result also deducible from (24) and the value of

$$(106) \qquad \Delta = \alpha\delta - \beta\gamma = \frac{1}{\sqrt{A}}.$$

We find further

$$
(107) \quad
\begin{aligned}
\sqrt{A}(P_{11} - P_{22}) &= \frac{2}{a}[a_{12}(a_{21})_u - a_{21}(a_{12})_u] \\
&\qquad + (a_{11} - a_{22})(p_{11} - p_{22}) + 2(a_{21}p_{12} + a_{12}p_{21}).
\end{aligned}
$$

These formulas assume a more compact form if we introduce the quantities $a_{ik}^{(1)}$ instead of the partial derivatives $(a_{ik})_u$. We find

$$
(108)\quad
\begin{aligned}
P_{11} &= p_{11} - \frac{A_u}{2A} + \frac{p_{12}\,a_{21} + p_{21}\,a_{12}}{2a} + \frac{a_{12}\,a_{21}^{(1)} - a_{21}\,a_{12}^{(1)}}{2a\sqrt{A}},\\
P_{22} &= p_{22} - \frac{A_u}{2A} - \frac{p_{12}\,a_{21} + p_{21}\,a_{12}}{2a} - \frac{a_{12}\,a_{21}^{(1)} - a_{21}\,a_{12}^{(1)}}{2a\sqrt{A}},\\
P_{12} &= -\frac{a_{12}^{(1)}}{\sqrt{A}} + \frac{a_{12}}{2a}\left[\frac{A_u}{A} + \frac{a_{11}^{(1)} - a_{22}^{(1)}}{\sqrt{A}}\right],\\
P_{21} &= +\frac{a_{21}^{(1)}}{\sqrt{A}} - \frac{a_{21}}{2a}\left[\frac{A_u}{A} + \frac{a_{11}^{(1)} - a_{22}^{(1)}}{\sqrt{A}}\right].
\end{aligned}
$$

On account of the frequently noted cogredience relations between the p_{ik}'s and the π_{ik}'s, we have similarly

$$
(109)\quad
\begin{aligned}
\Pi_{11} &= \pi_{11} - \frac{A_v}{2A} + \frac{\pi_{12}\,a_{21} + \pi_{21}\,a_{12}}{2a} + \frac{a_{12}\,a_{21}^{(2)} - a_{21}\,a_{12}^{(2)}}{2a\sqrt{A}},\\
\Pi_{22} &= \pi_{22} - \frac{A_v}{2A} - \frac{\pi_{12}\,a_{21} + \pi_{21}\,a_{12}}{2a} - \frac{a_{12}\,a_{21}^{(2)} - a_{21}\,a_{12}^{(2)}}{2a\sqrt{A}},\\
\Pi_{12} &= -\frac{a_{12}^{(2)}}{\sqrt{A}} + \frac{a_{12}}{2a}\left[\frac{A_v}{A} + \frac{a_{11}^{(2)} - a_{22}^{(2)}}{\sqrt{A}}\right],\\
\Pi_{21} &= +\frac{a_{21}^{(2)}}{\sqrt{A}} - \frac{a_{21}}{2a}\left[\frac{A_v}{A} + \frac{a_{11}^{(2)} - a_{22}^{(2)}}{\sqrt{A}}\right].
\end{aligned}
$$

The remaining coefficients, B_{ik} and Q_{ik}, of the focal canonical form are expressible in terms of those already computed by means of the formulas

$$
(110)\quad
\begin{aligned}
2B_{11} &= A_{11}\,P_{11} - \Pi_{11}, \qquad & 2B_{12} &= A_{11}\,P_{12} - \Pi_{12},\\
2B_{21} &= A_{22}\,P_{21} - \Pi_{21}, \qquad & 2B_{22} &= A_{22}\,P_{22} - \Pi_{22},
\end{aligned}
$$

and

$$
(111)\quad
\begin{aligned}
4Q_{11} &= 2(P_{11})_u + P_{11}^2 + P_{12}\,P_{21} - U_{11},\\
4Q_{12} &= 2(P_{12})_u + P_{12}(P_{11} + P_{22}) - U_{12},\\
4Q_{21} &= 2(P_{21})_u + P_{21}(P_{11} + P_{22}) - U_{21},\\
4Q_{22} &= 2(P_{22})_u + P_{22}^2 + P_{12}\,P_{21} - U_{22}.
\end{aligned}
$$

10. Determination of the developables of the congruence

The first order equations of system (S) were given originally in the form

$$
(112)\quad
\begin{aligned}
y_v &= a_{11}\,y_u + a_{12}\,z_u + b_{11}\,y + b_{12}\,z,\\
z_v &= a_{21}\,y_u + a_{22}\,z_u + b_{21}\,y + b_{22}\,z.
\end{aligned}
$$

Let us transform these equations by putting

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v), \tag{113}$$

where ϕ and ψ are arbitrary functions of u and v. The geometrical effect of this transformation is to replace the net of ruled surfaces formed by u = const. and v = const., by any other net $\bar{u}$ = const. and $\bar{v}$ = const., formed from the lines of the same congruence.

As a result of (113) we find

$$y_u = \frac{\partial y}{\partial \bar{u}}\phi_u + \frac{\partial y}{\partial \bar{v}}\psi_u, \qquad y_v = \frac{\partial y}{\partial \bar{u}}\phi_v + \frac{\partial y}{\partial \bar{v}}\psi_v, \text{ etc.,} \tag{114}$$

where we assume

$$\phi_u \psi_v - \phi_v \psi_u \neq 0. \tag{115}$$

Consequently (112) becomes

$$\begin{aligned} \frac{\partial y}{\partial \bar{v}} &= \bar{a}_{11}\frac{\partial y}{\partial \bar{u}} + \bar{a}_{12}\frac{\partial z}{\partial \bar{u}} + \bar{b}_{11}\, y + \bar{b}_{12}\, z, \\ \frac{\partial z}{\partial \bar{v}} &= \bar{a}_{21}\frac{\partial y}{\partial \bar{u}} + \bar{a}_{22}\frac{\partial z}{\partial \bar{u}} + \bar{b}_{21}\, y + \bar{b}_{22}\, z, \end{aligned} \tag{116}$$

where

$$\begin{aligned} \Delta\bar{a}_{11} &= -(a_{11}a_{22} - a_{12}a_{21})\phi_u\psi_u + a_{11}\phi_u\psi_v + a_{22}\phi_v\psi_u - \phi_v\psi_v, \\ \Delta\bar{a}_{22} &= -(a_{11}a_{22} - a_{12}a_{21})\phi_u\psi_u + a_{22}\phi_u\psi_v + a_{11}\phi_v\psi_u - \phi_v\psi_v, \\ \Delta\bar{a}_{12} &= a_{12}(\phi_u\psi_v - \phi_v\psi_u), \qquad \Delta\bar{a}_{21} = a_{21}(\phi_u\psi_v - \phi_v\psi_u), \\ \Delta\bar{b}_{11} &= (a_{12}b_{21} - a_{22}b_{11})\psi_u + b_{11}\psi_v, \\ \Delta\bar{b}_{12} &= (a_{12}b_{22} - a_{22}b_{12})\psi_u + b_{12}\psi_v, \\ \Delta\bar{b}_{21} &= (a_{21}b_{11} - a_{11}b_{21})\psi_u + b_{21}\psi_v, \\ \Delta\bar{b}_{22} &= (a_{21}b_{12} - a_{11}b_{22})\psi_u + b_{22}\psi_v, \\ \Delta &= (a_{11}a_{22} - a_{12}a_{21})\psi_u^2 - (a_{11} + a_{22})\psi_u\psi_v + \psi_v^2. \end{aligned} \tag{117}$$

As long as Δ is different from zero, the equations (116) are of the same form as (112). If $\Delta = 0$, the two equations (116) reduce to an equation of the form

$$\alpha\frac{\partial y}{\partial \bar{u}} + \beta\frac{\partial z}{\partial \bar{u}} + \gamma y + \delta z = 0,$$

showing that the ruled surfaces $\bar{v}$ = const. are developables.*

If $A = (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})$ is different from zero, the condition $\Delta = 0$ can be satisfied in two essentially distinct ways, by integrating the differential equations for ψ which are obtained by equating to zero the

* *Proj. Diff. Geom.*, p. 131.

factors of Δ. This proves the familiar theorem that a congruence with distinct focal surfaces has two distinct one-parameter families of developables.

If $A = a_1^2 - 4a_2$ is equal to zero, the two sheets of the focal surface coincide, and there exists only a single one-parameter family of developables in the congruence. Let us assume that the surface of reference S_y is so chosen as to coincide with the focal surface. Then we shall have

$$a_{12} = 0, \qquad a_{11} - a_{22} = 0, \qquad a_{21} \neq 0, \tag{118}$$

since the covariant

$$a_{21}\, y^2 - (a_{11} - a_{22})\, yz - a_{12}\, z^2$$

must reduce to a multiple of y^2. Then Δ is a perfect square, and the differential equation $\Delta = 0$ reduces to

$$a_{11}\, \psi_u - \psi_v = 0. \tag{119}$$

If $\psi(u, v)$ is a solution of this equation, the ruled surfaces $\psi(u, v) = \text{const.}$ will be the developables of the congruence.

On the other hand, if we assume $b_{12} \neq 0$, equations (17) show that the differential equation of the asymptotic lines on S_y is

$$- p_{12}\, du^2 + 2c_{12}\, dudv + e_{12}\, dv^2 = 0, \tag{120}$$

and, in our case, we have

$$c_{12} = - a_{11}\, p_{12} + b_{12}, \qquad e_{12} = a_{11}\, c_{12} + b_{12}\, a_{11} = - a_{11}^2\, p_{12} + 2a_{11}\, b_{12}.$$

Consequently (120) becomes

$$(du + a_{11}\, dv)\, [\, -p_{12}\, (du + a_{11}\, dv) + 2b_{12}\, dv] = 0. \tag{121}$$

But an integral of

$$du + a_{11}\, dv = 0$$

is a solution of (119), giving rise to the familiar theorem that a congruence with coincident focal surfaces is composed of the tangents of one of the two families of asymptotic lines on its focal surface. The assumption $b_{12} \neq 0$ upon which we have based the proof, merely requires that the focal surface do not degenerate into a curve. Equation (121) makes it convenient to examine the second set of asymptotic lines of the focal surface and the corresponding congruence of tangents.

If the two focal sheets coincide and at the same time degenerate into coincident curves, we may assume

$$a_{11} = 0, \qquad a_{12} = 0, \qquad a_{22} = 0, \qquad a_{21} \neq 0, \qquad b_{12} = 0$$

if we use the coincident focal curves for the locus of P_y, and the developables of the congruence as surfaces $u = \text{const.}$ We shall then have

$$y_v = b_{11} y, \qquad z_v = b_{21} y + b_{22} z.$$

These equations show that P_y remains fixed when v alone changes, and that P_z can only move along the straight line joining it to P_y when v alone changes. If v remains constant, and u only changes, P_y moves on the double focal curve F, and the plane whose homogeneous coördinates are given by the four third order determinants of the form

$$\lambda = |y, z, y_u|$$

contains P_z and the tangent of F at P_y. We find

$$\lambda_v = (2b_{11} + b_{22})\lambda.$$

Consequently this plane does not change when v changes; it changes only with u. Consequently with the variation of both variables, this plane generates a developable which contains the curve F. The congruence may therefore be described as follows. On an arbitrary developable draw any curve. The lines tangent to the developable along this curve constitute a congruence of this sort. Clearly the developables of this congruence are the ∞^1 pencils whose vertices are the points of the given curve, and whose planes are the tangent planes of the fundamental developable.

There remains one case which requires special attention. If

$$a_{12} = a_{21} = a_{11} - a_{22} = 0,$$

the foci are entirely indeterminate, but the developables are not, and we may assume

$$a_{11} = a_{22} = a_{12} = a_{21} = 0.$$

The equations

$$y_v = b_{11} y + b_{12} z, \qquad z_v = b_{21} y + b_{22} z$$

show that P_y and P_z move along the same fixed straight line if v varies while u remains constant. Thus the congruence, in this case degenerates into a one-parameter family of straight lines. All of the ruled surfaces of the family $v =$ const. coincide.

11. The differential equations of the congruence referred to its focal sheets and developables

We cannot introduce the developables of the congruence as parametric ruled surfaces and also preserve the form (S) of our system of differential equations. We now proceed to show how (S) will be transformed.

Let us start with the first order equations of (S) written in the focal canonical form

$$\begin{aligned} \bar{y}_v &= A_{11} \bar{y}_u + B_{11} \bar{y} + B_{12} \bar{z}, \\ \bar{z}_v &= A_{22} \bar{z}_u + B_{21} \bar{y} + B_{22} \bar{z}, \end{aligned} \qquad \sqrt{A} = A_{11} - A_{22} \neq 0, \tag{122}$$

so that $S_{\bar{y}}$ and $S_{\bar{z}}$ are the focal sheets, assumed to be distinct, but not necessarily non-degenerate. The expressions for A_{ik} and B_{ik} in terms of the coefficients of (S) are given in Article 9.

Since $A_{12} = A_{21} = 0$, the developables of the congruence are obtained by equating to constants the functions

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v), \tag{123}$$

where ϕ and ψ satisfy the conditions

$$\phi_v = A_{11}\,\phi_u, \qquad \psi_v = A_{22}\,\psi_u, \tag{124}$$

respectively, and where constant solutions of these equations are, of course, excluded. We may, therefore, assume $\phi_u \neq 0$, $\psi_u \neq 0$.

If we introduce $\bar{u}$ and $\bar{v}$, as defined by (123) and (124), as independent variables, the equations (122) assume the form

$$\frac{\partial \bar{y}}{\partial \bar{v}} = m'\,\bar{y} + m''\,\bar{z}, \qquad \frac{\partial \bar{z}}{\partial \bar{u}} = n'\,\bar{y} + n''\,\bar{z}, \tag{125}$$

where

$$m' = -\frac{\beta_{11}}{\psi_u}, \qquad m'' = -\frac{\beta_{12}}{\psi_u}, \qquad n' = \frac{\beta_{21}}{\phi_u}, \qquad n'' = \frac{\beta_{22}}{\phi_u}, \qquad \beta_{ik} = \frac{B_{ik}}{\sqrt{A}}. \tag{126}$$

A transformation of the form

$$\bar{y} = \omega\eta, \qquad \bar{z} = \omega'\zeta \tag{127}$$

will not change the surfaces of reference. Its effect upon the fundamental quantities entering into the focal canonical form is given by the equations

$$\begin{aligned}
&\bar{P}_{11} = P_{11} + 2\frac{\omega_u}{\omega}, \qquad \bar{P}_{12} = \frac{\omega'}{\omega}P_{12}, \\
&P_{21} = \frac{\omega}{\omega'}P_{21}, \qquad \bar{P}_{22} = P_{22} + 2\frac{\omega'_u}{\omega'}, \\
&\bar{U}_{11} = U_{11}, \qquad \bar{U}_{12} = \frac{\omega'}{\omega}U_{12}, \qquad \bar{U}_{21} = \frac{\omega}{\omega'}U_{21}, \qquad \bar{U}_{22} = U_{22}, \\
&\bar{A}_{11} = A_{11}, \qquad \bar{A}_{12} = 0, \qquad \bar{A}_{21} = 0, \qquad \bar{A}_{22} = A_{22}, \\
&\bar{\Pi}_{11} = \Pi_{11} + 2\frac{\omega_v}{\omega}, \qquad \bar{\Pi}_{12} = \frac{\omega'}{\omega}\Pi_{12}, \\
&\bar{\Pi}_{21} = \frac{\omega}{\omega'}\Pi_{21}, \qquad \bar{\Pi}_{22} = \Pi_{22} + 2\frac{\omega'_v}{\omega'}, \\
&\bar{B}_{11} = B_{11} + A_{11}\frac{\omega_u}{\omega} - \frac{\omega_v}{\omega}, \qquad \bar{B}_{12} = \frac{\omega'}{\omega}B_{12}, \\
&\bar{B}_{21} = \frac{\omega}{\omega'}B_{21}, \qquad \bar{B}_{22} = B_{22} + A_{22}\frac{\omega'_u}{\omega'} - \frac{\omega'_v}{\omega'}.
\end{aligned} \tag{128}$$

The equations for $\bar{B}_{11}$ and $\bar{B}_{22}$ show that these quantities may be equated to zero provided that ω and ω' are chosen as solutions of the equations

$$(129)\qquad A_{11}\frac{\omega_u}{\omega} - \frac{\omega_v}{\omega} + B_{11} = 0, \qquad A_{22}\frac{\omega'_u}{\omega'} - \frac{\omega'_v}{\omega'} + B_{22} = 0$$

respectively. If we apply the transformation (127) directly to equations (126) we find the conditions

$$(130)\qquad \frac{\partial \log \omega}{\partial \bar{v}} = -\frac{B_{11}}{\psi_u\sqrt{A}} = m', \qquad \frac{\partial \log \omega'}{\partial \bar{u}} = \frac{+B_{22}}{\phi_u\sqrt{A}} = n''$$

equivalent to (129). Consequently this simplification of (125), resulting in equations of the form

$$(131)\qquad \frac{\partial \eta}{\partial \bar{v}} = m\zeta, \qquad \frac{\partial \zeta}{\partial \bar{u}} = n\eta,$$

where

$$(132)\qquad m = -\frac{\omega'}{\omega}\frac{B_{12}}{\psi_u\sqrt{A}} = \frac{\omega'}{\omega}m'', \qquad n = \frac{\omega}{\omega'}\frac{B_{21}}{\phi_u\sqrt{A}} = \frac{\omega}{\omega'}n',$$

may be accomplished by quadratures after the differential equations (124) which determine the developables of the congruence have been integrated.

The second order differential equations of (S) in its focal canonical form are

$$(133)\qquad \begin{aligned} \bar{y}_{uu} + P_{11}\bar{y}_u + P_{12}\bar{z}_u + Q_{11}\bar{y} + Q_{12}\bar{z} &= 0, \\ \bar{z}_{uu} + P_{21}\bar{y}_u + P_{22}\bar{z}_u + Q_{21}\bar{y} + Q_{22}\bar{z} &= 0. \end{aligned}$$

Let us introduce the variables $\bar{u}$ and $\bar{v}$ into these equations. We have

$$(134)\qquad \begin{aligned} \bar{y}_u &= \bar{y}_{\bar{u}}\phi_u + \bar{y}_{\bar{v}}\psi_u = \bar{y}_{\bar{u}}\phi_u + (m'\bar{y} + m''\bar{z})\psi_u = \bar{y}_{\bar{u}}\phi_u - \frac{B_{11}\bar{y} + B_{12}\bar{z}}{\sqrt{A}}, \\ \bar{z}_u &= \bar{z}_{\bar{u}}\phi_u + \bar{z}_{\bar{v}}\psi_u = (n'\bar{y} + n''\bar{z})\phi_u + \bar{z}_{\bar{v}}\psi_u = \bar{z}_{\bar{v}}\psi_u + \frac{B_{21}\bar{y} + B_{22}\bar{z}}{\sqrt{A}}, \end{aligned}$$

where we have made use of (125) and (126).

From (125) we find by differentiation

$$(135)\qquad \begin{aligned} \bar{y}_{\bar{u}\bar{v}} &= m'\bar{y}_{\bar{u}} + (m'_{\bar{u}} + m''n')\bar{y} + (m''_{\bar{u}} + m''n'')\bar{z}, \\ \bar{z}_{\bar{u}\bar{v}} &= n''\bar{z}_{\bar{v}} + (n'_{\bar{v}} + n'm')\bar{y} + (n''_{\bar{v}} + n'm'')\bar{z}. \end{aligned}$$

If F denotes an arbitrary function of u and v, we have

$$(136a)\qquad F_u = F_{\bar{u}}\phi_u + F_{\bar{v}}\psi_u, \qquad F_v = A_{11}F_{\bar{u}}\phi_u + A_{22}F_{\bar{v}}\psi_u,$$

and

$$(136b)\qquad F_{\bar{u}} = \frac{1}{\phi_u}U(F), \qquad F_{\bar{v}} = \frac{1}{\psi_u}V(F),$$

where

$$(136c)\qquad U(F) = \frac{1}{\sqrt{A}}(-A_{22}F_u + F_v), \qquad V(F) = \frac{1}{\sqrt{A}}(A_{11}F_u - F_v).$$

The use of these formulas gives us the following equations for the coefficients of $\bar{y}$ and $\bar{z}$ in (135):

$$
(137) \qquad
\begin{aligned}
\phi_u \psi_u (m'_{\bar{u}} + m'' n') &= -\beta_{12}\beta_{21} + \beta_{11}\alpha_{22} - U(\beta_{11}),\\
\phi_u \psi_u (m''_{\bar{u}} + m'' n'') &= -\beta_{12}\beta_{22} + \beta_{12}\alpha_{22} - U(\beta_{12}),\\
\phi_u \psi_u (n'_{\bar{v}} + m' n') &= -\beta_{11}\beta_{21} + \beta_{21}\alpha_{11} + V(\beta_{21}),\\
\phi_u \psi_u (n''_{\bar{v}} + m'' n') &= -\beta_{12}\beta_{21} + \beta_{22}\alpha_{11} + V(\beta_{22}),
\end{aligned}
$$

where

$$
(137a) \qquad \alpha_{11} = \frac{(A_{11})_u}{\sqrt{A}}, \qquad \alpha_{22} = \frac{(A_{22})_u}{\sqrt{A}}.
$$

From (134) we finu

$$
\begin{aligned}
\bar{y}_{uu} = \phi_u [\bar{y}_{\bar{u}\bar{u}}\phi_u + \bar{y}_{\bar{u}\bar{v}}\psi_u] + y_{\bar{u}}\phi_{uu} - (\beta_{11})_u \bar{y} - (\beta_{12})_u \bar{z}\\
- \beta_{11}(\bar{y}_{\bar{u}}\phi_u + \bar{y}_{\bar{v}}\psi_u) - \beta_{12}(\bar{z}_{\bar{u}}\phi_u + \bar{z}_{\bar{v}}\psi_u).
\end{aligned}
$$

We substitute, into this equation, the values (125) of $\bar{y}_{\bar{v}}$ and $\bar{z}_{\bar{u}}$, and the value of $\bar{y}_{\bar{u}\bar{v}}$ from (135) and (137). The resulting expression is

$$
\begin{aligned}
\bar{y}_{uu} = \phi_u^2 \bar{y}_{\bar{u}\bar{u}} + (\phi_{uu} - 2\beta_{11}\phi_u)\bar{y}_{\bar{u}} - \beta_{12}\psi_u \bar{z}_{\bar{v}}\\
+ [\beta_{11}^2 - 2\beta_{12}\beta_{21} + \beta_{11}\alpha_{22} - (\beta_{11})_u - U(\beta_{11})]\bar{y}\\
+ [\beta_{11}\beta_{12} - 2\beta_{12}\beta_{22} + \beta_{12}\alpha_{22} - (\beta_{12})_u - U(\beta_{12})]\bar{z}.
\end{aligned}
$$

Finally we substitute this expression in (133), and transform the second equation of (133) in similar fashion. We obtain the following differential equations

$$
(138) \qquad
\begin{aligned}
\bar{y}_{\bar{u}\bar{u}} &= \bar{a}\bar{y} + \bar{b}\bar{z} + \bar{c}\bar{y}_{\bar{u}} + \bar{d}\bar{z}_{\bar{v}},\\
\bar{z}_{\bar{v}\bar{v}} &= \bar{a}'\bar{y} + \bar{b}'\bar{z} + \bar{c}'\bar{y}_{\bar{u}} + \bar{d}'\bar{z}_{\bar{v}},
\end{aligned}
$$

where

$$
(139) \qquad
\begin{aligned}
\phi_u^2 \bar{a} &= -Q_{11} + \beta_{11}P_{11} - \beta_{21}P_{12} - \beta_{11}^2 + 2\beta_{12}\beta_{21}\\
&\qquad + (\beta_{11})_u - \alpha_{22}\beta_{11} + U(\beta_{11}),\\
\phi_u^2 \bar{b} &= -Q_{12} + \beta_{12}P_{11} - \beta_{22}P_{12} - \beta_{11}\beta_{12} + 2\beta_{12}\beta_{22}\\
&\qquad + (\beta_{12})_u - \alpha_{22}\beta_{12} + U(\beta_{12}),\\
\phi_u \bar{c} &= -P_{11} + 2\beta_{11} - \frac{\phi_{uu}}{\phi}, \qquad \phi_u^2 \bar{d} = \psi_u(-P_{12} + \beta_{12}),\\
\psi_u^2 \bar{a}' &= -Q_{21} + \beta_{11}P_{21} - \beta_{21}P_{22} + 2\beta_{11}\beta_{21} - \beta_{21}\beta_{22}\\
&\qquad - (\beta_{21})_u - \alpha_{11}\beta_{21} - V(\beta_{21}),\\
\psi_u^2 \bar{b}' &= -Q_{22} + \beta_{12}P_{21} - \beta_{22}P_{22} - \beta_{22}^2 + 2\beta_{12}\beta_{21}\\
&\qquad - (\beta_{22})_u - \alpha_{11}\beta_{22} - V(\beta_{22}),\\
\psi_u^2 \bar{c}' &= \phi_u(-P_{21} - \beta_{21}), \qquad \psi_u \bar{d}' = -P_{22} - 2\beta_{22} - \frac{\psi_{uu}}{\psi_u},
\end{aligned}
$$

and where it is desirable to remember that

$$\sqrt{A} = A_{11} - A_{22}. \tag{140}$$

Finally, if we apply the transformation (127) to (138), we obtain the equations

$$\begin{aligned} \frac{\partial^2 \eta}{\partial \bar{u}^2} &= a\eta + b\zeta + c\frac{\partial \eta}{\partial \bar{u}} + d\frac{\partial \zeta}{\partial \bar{v}}, \\ \frac{\partial^2 \zeta}{\partial \bar{v}^2} &= a'\eta + b'\zeta + c'\frac{\partial \eta}{\partial \bar{u}} + d'\frac{\partial \zeta}{\partial \bar{v}}, \end{aligned} \tag{141}$$

where

$$\begin{gathered} a = \bar{a} + \bar{c}\frac{\omega_{\bar{u}}}{\omega} - \frac{\omega_{\bar{u}\bar{u}}}{\omega}, \qquad b = \frac{\omega'}{\omega}\left(\bar{b} + \frac{\omega'_{\bar{v}}}{\omega'}\bar{d}\right), \\ c = \bar{c} - 2\frac{\omega_{\bar{u}}}{\omega}, \qquad d = \frac{\omega'}{\omega}\bar{d}, \\ a' = \frac{\omega}{\omega'}\left(\bar{a}' + \frac{\omega_{\bar{u}}}{\omega}\bar{c}'\right), \qquad b' = \bar{b}' + \frac{\omega'_{\bar{v}}}{\omega'}\bar{d}' - \frac{\omega'_{\bar{v}\bar{v}}}{\omega'}, \\ c' = \frac{\omega}{\omega'}\bar{c}', \qquad d' = \bar{d}' - 2\frac{\omega'_{\bar{v}}}{\omega'}, \end{gathered} \tag{142}$$

and where ω and ω' satisfy the conditions (130). *Equations* (131) *and* (141) *together are the differential equations of the congruence in the canonical form,* and all of the invariants and covariants of the congruence can now be expressed in terms of the coefficients and variables of the original system (S). The functions ϕ_u, ψ_u, ω, and ω', whose actual values are required for the purpose of computing the coefficients m, n, a, b, c, d, etc., enter into the expressions for these invariants and covariants only as extraneous factors, and are eliminated entirely from the expressions for *absolute* invariants. Thus we have at once the expressions

$$\begin{gathered} m = -\frac{\omega'}{\omega\psi_u}\beta_{12}, \qquad n = \frac{\omega}{\omega'\phi_u}\beta_{21}, \\ c' = -\frac{\omega\phi_u}{\omega'\psi_u^2}(P_{21} + \beta_{21}), \qquad d = -\frac{\omega'\psi_u}{\omega\phi_u^2}(P_{12} - \beta_{12}), \\ W = mn - c'd = \frac{1}{\phi_u\psi_u}(-P_{12}P_{21} + \beta_{12}P_{21} - \beta_{21}P_{12}), \end{gathered} \tag{143}$$

for the five most fundamental invariants of the congruence. If m or n vanishes one of the focal sheets degenerates into a curve; if c' or d vanishes, one of the focal sheets is a developable. If $W = 0$, the congruence is a W-congruence.

In order to find the expressions for the invariants $\overset{(\eta)}{\mathfrak{B}}$, $\overset{(\zeta)}{\mathfrak{B}}$, $\overset{(\eta)}{\mathfrak{C}}{}''$, and $\overset{(\zeta)}{\mathfrak{C}}{}''$,* the following formulas will be convenient:

* Brussels Paper, p. 20.

$$
(144)\qquad
\begin{aligned}
&U \log \omega = \beta_{11} + \frac{\omega_u}{\omega}, && U \log \omega' = \beta_{22},\\
&U \log \phi_u = \alpha_{11} + \frac{\phi_{uu}}{\phi_u}, && U \log \psi_u = \alpha_{22},\\
&V \log \omega = -\beta_{11}, && V \log \omega' = -\beta_{22} + \frac{\omega'_u}{\omega'},\\
&V \log \phi_u = -\alpha_{11}, && V \log \psi_u = -\alpha_{22} + \frac{\psi_{uu}}{\psi_u}.
\end{aligned}
$$

We find

$$
4\overset{(\eta)}{\mathfrak{B}} = c - \frac{1}{2}\frac{d_{\bar{u}}}{d} - \frac{3}{2}\frac{m_{\bar{u}}}{m} = \bar{c} - \frac{1}{2}\frac{\partial \log dm^3\omega^4}{\partial \bar{u}}.
$$

But from (143) we have

$$
dm^3\omega^4 = \frac{(\omega')^4}{\phi_u^2\psi_u^2}\beta_{12}^3(P_{12} - \beta_{12}).
$$

Consequently, making use of (136*b*), and (144), we find

$$
(145a)\quad 4\phi_u\overset{(\eta)}{\mathfrak{B}} = \alpha_{11} + \alpha_{22} + 2(\beta_{11} - \beta_{22}) - P_{11} - \tfrac{1}{2}U \log \beta_{12}^3(P_{12} - \beta_{12}).
$$

Again we have

$$
4\overset{(\zeta)}{\mathfrak{B}} = c + \frac{3}{2}\frac{c'_{\bar{u}}}{c'} + \frac{1}{2}\frac{n_{\bar{u}}}{n} = \bar{c} + \frac{1}{2}\frac{\partial \log \dfrac{(c')^3 n}{\omega^4}}{\partial \bar{u}},
$$

and

$$
\frac{(c')^3 n}{\omega^4} = -\frac{\phi_u^2}{(\omega')^4\psi_u^6}\beta_{21}(P_{21} + \beta_{21})^3,
$$

whence

$$
(145b)\quad 4\phi_u\overset{(\zeta)}{\mathfrak{B}} = \alpha_{11} - 3\alpha_{22} + 2(\beta_{11} - \beta_{22}) - P_{11} + \tfrac{1}{2}U \log \beta_{21}(P_{21} + \beta_{21})^3.
$$

In the same way we find

$$
(145c)\quad 4\psi_u\overset{(\eta)}{\mathfrak{C}}{}'' = 3\alpha_{11} - \alpha_{22} + 2(\beta_{11} - \beta_{22}) - P_{22} + \tfrac{1}{2}V \log \beta_{12}(P_{12} - \beta_{12})^3,
$$

and

$$
(145d)\quad 4\psi_u\overset{(\zeta)}{\mathfrak{C}}{}'' = -\alpha_{11} - \alpha_{22} + 2(\beta_{11} - \beta_{22}) - P_{22} - \tfrac{1}{2}V \log \beta_{21}^3(P_{21} + \beta_{21}).
$$

The values of

$$
(146)\qquad
\begin{aligned}
\theta^{(\eta)} &= \frac{2}{\sqrt{-dm}}\left(m\overset{(\eta)}{\mathfrak{B}}{}^2 + d\overset{(\eta)}{\mathfrak{C}}{}''^2\right),\\
\theta^{(\zeta)} &= \frac{2}{\sqrt{-c'n}}\left(c'\overset{(\zeta)}{\mathfrak{B}}{}^2 + n\overset{(\zeta)}{\mathfrak{C}}{}''^2\right)
\end{aligned}
$$

can now easily be computed. These invariants vanish if and only if the focal sheet $S^{(\eta)}$, or $S^{(\xi)}$, or both, are ruled surfaces.

The Laplace-Darboux invariants of the conjugate systems on S_η and S_ζ are

$$h = mn, \qquad k = mn - \frac{\partial^2 \log m}{\partial \bar{u} \partial \bar{v}},*$$

and

(147a) $$h_{-1} = mn - \frac{\partial^2 \log n}{\partial \bar{u} \partial \bar{v}}, \qquad k_{-1} = mn = h.$$

We find at once

(147b) $$h = k_{-1} = -\frac{\beta_{12}\beta_{21}}{\phi_u \psi_u}.$$

To obtain symmetric expressions for k and h_{-1}, we observe that, from (136b), we may compute $F_{\bar{u}\bar{v}}$ in two ways, giving

$$\phi_u \psi_u F_{\bar{u}\bar{v}} = VU(F) + \alpha_{11} U(F) = UV(F) - \alpha_{22} V(F),$$

so that we find the identity

(148) $$VU(F) - UV(F) + \alpha_{11} U(F) + \alpha_{22} V(F) = 0,$$

and the symmetrical expression for $F_{\bar{u}\bar{v}}$

(149) $$2\phi_u \psi_u F_{\bar{u}\bar{v}} = UV(F) + VU(F) + \alpha_{11} U(F) - \alpha_{22} V(F).$$

We propose to apply this formula for $F = \log m$ and for $F = \log n$. We find first

(150)
$$U \log m = -\alpha_{22} - \beta_{11} + \beta_{22} + U \log \beta_{12} - \frac{\omega_u}{\omega},$$
$$V \log m = +\alpha_{22} + \beta_{11} - \beta_{22} + V \log \beta_{12} + \frac{\omega_u'}{\omega'} - \frac{\psi_{uu}}{\psi_u},$$
$$U \log n = -\alpha_{11} + \beta_{11} - \beta_{22} + U \log \beta_{21} + \frac{\omega_u}{\omega} - \frac{\phi_{uu}}{\phi_u},$$
$$V \log n = +\alpha_{11} - \beta_{11} + \beta_{22} + V \log \beta_{21} - \frac{\omega_u'}{\omega'}.$$

Let us note further that

(151)
$$U(\omega_u') = (\beta_{22})_u \omega' + \frac{1}{2}\frac{A_u}{A}\beta_{22}\omega' + (\alpha_{22} + \beta_{22})\omega_u',$$
$$U(\psi_{uu}) = (\alpha_{22})_u \psi_u + \frac{1}{2}\frac{A_u}{A}\alpha_{22}\psi_u + 2\alpha_{22}\psi_{uu},$$
$$V(\omega_u) = -(\beta_{11})_u \omega - \frac{1}{2}\frac{A_u}{A}\beta_{11}\omega - (\alpha_{11} + \beta_{11})\omega,$$
$$V(\phi_{uu}) = -(\alpha_{11})_u \phi_u - \frac{1}{2}\frac{A_u}{A}\alpha_{11}\phi_u - 2\alpha_{11}\phi_{uu}.$$

* Darboux, *Théorie des surfaces*, vol. 2, and E. J. Wilczynski, *The general theory of congruences*, these Transactions, vol. 16 (1915), p. 318.

Tben we find, from (149), (150), and (151),

(152)
$$
\begin{aligned}
2\phi_u \psi_u \frac{\partial^2 \log m}{\partial \bar{u} \partial \bar{v}} = {} & UV \log \beta_{12} + VU \log \beta_{12} + \alpha_{11} U \log \beta_{12} \\
& - \alpha_{22} V \log \beta_{12} + U(\alpha_{22} + \beta_{11} - \beta_{22}) \\
& - V(\alpha_{22} + \beta_{11} - \beta_{22}) + (\beta_{11} + \beta_{22} - \alpha_{22})_u \\
& + \frac{1}{2} \frac{A_u}{A} (\beta_{11} + \beta_{22} - \alpha_{22}) \\
& - (\alpha_{11} + \alpha_{22})(\alpha_{22} + \beta_{11} - \beta_{22}), \\
2\phi_u \psi_u \frac{\partial^2 \log n}{\partial \bar{u} \partial \bar{v}} = {} & UV \log \beta_{21} + VU \log \beta_{21} + \alpha_{11} U \log \beta_{21} \\
& - \alpha_{22} V \log \beta_{21} + U(\alpha_{11} - \beta_{11} + \beta_{22}) \\
& - V(\alpha_{11} - \beta_{11} + \beta_{22}) - (\beta_{11} + \beta_{22} - \alpha_{11})_u \\
& - \frac{1}{2} \frac{A_u}{A} (\beta_{11} + \beta_{22} - \alpha_{11}) \\
& - (\alpha_{11} + \alpha_{22})(\alpha_{11} - \beta_{11} + \beta_{22}).
\end{aligned}
$$

These expressions may be written in somewhat shorter, but less symmetric form, by making use of (148). The complete expressions for k and h_{-1} follow at once from (147) and (152).

The integrability conditions of the system, composed of (131) and (141), require the existence of a function f, of $\bar{u}$ and $\bar{v}$, such that

(153)
$$c = \frac{\partial f}{\partial \bar{u}}, \qquad d' = \frac{\partial f}{\partial \bar{v}}.$$

We wish to verify this directly and, at the same time, obtain the relation between this function and the function P, of u and v, whose partial derivatives are

(154)
$$P_u = P_{11} + P_{22}, \qquad P_v = \Pi_{11} + \Pi_{22} - (A_{11} + A_{22})_u,$$

and whose existence is assured as a result of one of the integrability conditions of our system (S), in its focal canonical form. We have

$$c = \bar{c} - 2\frac{\partial \log \omega}{\partial \bar{u}}, \qquad d' = \bar{d}' - 2\frac{\partial \log \omega'}{\partial \bar{v}}.$$

Consequently the condition $c_{\bar{v}} = d'_{\bar{u}}$, which is equivalent to (153), may be written in the form

(155)
$$\left(\bar{c} + 2\frac{B_{22}}{\phi_u \sqrt{A}}\right)_{\bar{v}} = \left(\bar{d}' - 2\frac{B_{11}}{\psi_u \sqrt{A}}\right)_{\bar{u}}.$$

But we find, making use of (110), (136c), (139), and (140), that

$$\bar{c} + \frac{2B_{22}}{\phi_u \sqrt{A}} = -\frac{1}{\phi_u}\left[\frac{-A_{22}P_u + P_v + (A_{11}+A_{22})_u}{\sqrt{A}} + \frac{\phi_{uu}}{\phi_u}\right]$$

$$= -P_{\bar{u}} - \frac{1}{\phi_u}\left[\frac{(A_{11}+A_{22})_u}{\sqrt{A}} + \frac{\phi_{uu}}{\phi_u}\right],$$

$$\bar{d}' - \frac{2B_{11}}{\psi_u \sqrt{A}} = -\frac{1}{\psi_u}\left[\frac{A_{11}P_u - P_v - (A_{11}+A_{22})_u}{\sqrt{A}} + \frac{\psi_{uu}}{\psi_u}\right]$$

$$= -P_{\bar{v}} - \frac{1}{\psi_u}\left[\frac{-(A_{11}+A_{22})_u}{\sqrt{A}} + \frac{\psi_{uu}}{\psi_u}\right].$$

These expressions will satisfy (155) if and only if there exists a function F whose partial derivatives with respect to $\bar{u}$ and $\bar{v}$ can be identified with the bracket terms of the two right members. We must have in that case

$$-A_{22}F_u + F_v = (A_{11}+A_{22})_u + \frac{\phi_{uu}}{\phi_u}(A_{11} - A_{22}),$$

$$+A_{11}F_u - F_v = -(A_{11}+A_{22})_u + \frac{\psi_{uu}}{\psi_u}(A_{11} - A_{22}),$$

whence

$$F_u = \frac{\phi_{uu}}{\phi_u} + \frac{\psi_{uu}}{\psi_u}, \qquad F_v = (A_{11}+A_{22})_u + A_{11}\frac{\phi_{uu}}{\phi_u} + A_{22}\frac{\psi_{uu}}{\psi_u}.$$

The consistency of these conditions may be verified directly as follows.

The first equation gives on integration

$$F = \log \phi_u \psi_u + V(v)$$

where $V(v)$ is a function of v alone. If we substitute this expression for F into the equation for F_v, and make use of the conditions

$$\phi_v = A_{11}\phi_u, \qquad \psi_v = A_{22}\psi_u$$

which define ϕ and ψ, we find $V' = 0$. Consequently we have found

$$\bar{c} + \frac{2B_{22}}{\phi_u \sqrt{A}} = -\frac{\partial}{\partial \bar{u}}(P + \log C\phi_u \psi_u),$$

$$\bar{d}' - \frac{2B_{11}}{\psi_u \sqrt{A}} = -\frac{\partial}{\partial \bar{v}}(P + \log C\phi_u \psi_u),$$

where C is a non-vanishing constant, and therefore

$$c = -\frac{\partial}{\partial \bar{u}}(P + \log C\phi_u \psi_u + 2\log \omega' + 2\log \omega) = \frac{\partial f}{\partial \bar{u}}$$

$$d' = -\frac{\partial}{\partial \bar{v}}(P + \log C\phi_u \psi_u + 2\log \omega + 2\log \omega') = \frac{\partial f}{\partial \bar{v}},$$

whence

$$f = -[P + \log (\omega\omega')^2 \phi_u \psi_u + k]$$

where k is a constant. According to (105) and (154) we have

$$P_u = p_u - \frac{A_u}{A}, \qquad P_v = p_v - \frac{A_v}{A}$$

and therefore

(156) $$f = -[p + \log A^{-1} \phi_u \psi_u (\omega\omega')^2 + k]$$

where p is defined by the consistent conditions

$$p_u = p_{11} + p_{22}, \qquad p_v = \pi_{11} + \pi_{22} - (a_{11} + a_{22})_u,$$

and where p is closely connected with the value of

$$D = |y_u, z_u, y, z|,$$

the principal determinant of system (S), this fourth order determinant being formed from four independent solutions of (S).

By using the differential equations of (S), we find

$$D_u = |y_{uu}, z_u, y, z| + |y_u, z_{uu}, y, z| = -(p_{11} + p_{22}) D$$

$$D_v = |y_{uv}, z_u, y, z| + |y, z_{uv}, y, z| + |y_u, z_u, y_v, z| + |y_u, z_u, y, z_v|$$

$$= (b_{11} + b_{22} + c_{11} + c_{22}) D_v = -[\pi_{11} + \pi_{22} - (a_{11} + a_{22})_u] D,$$

so that

(157) $$p_u = p_{11} + p_{22} = -\frac{D_u}{D}, \qquad p_v = \pi_{11} + \pi_{22} - (a_{11} + a_{22})_u = -\frac{D_v}{D}.$$

Consequently we must have

(158) $$D = Ce^{-p}$$

where C is a non-vanishing constant whose actual value will depend upon what particular solutions of (S) have been used in forming the determinant D. We may write finally

(159) $$f = \log AD\phi_u^{-1} \psi_u^{-1} (\omega\omega')^{-2} K$$

where K is a non-vanishing constant.

On the other hand we have also

(160) $$f = \log K' \Delta^*$$

where the fourth order determinant

(161) $$\Delta = |\eta, \zeta, \eta_{\bar{u}}, \zeta_{\bar{v}}|$$

is formed from the corresponding four solutions of the system (131), (141). The resulting relation between D and Δ may be verified directly.

* Brussels Paper, p. 26.

12. The differential equations of the net of ruled surfaces in planar coördinates and the principle of duality

Let us return to the differential equations (S) of the one parameter family of ruled surfaces. The generating lines of these surfaces have so far been regarded as the lines of junction of corresponding points, P_y and P_z, of the two surfaces of reference S_y and S_z. We propose now to think of these same lines as lines of intersections of corresponding planes.

Let R be a ruled surface of the family $v = \text{const.}$ The coördinates of the plane tangent to R at P_y are the four third order determinants of the form

$$r = |y, z, y_u|. \tag{162a}$$

Similarly the determinants of the form

$$s = |y, z, z_u| \tag{162b}$$

serve as coördinates for the plane tangent to R at P_z. We propose to obtain a system of differential equations similar to (S) which will be satisfied by the four pairs of functions (r_i, s_i), $(i = 1, 2, 3, 4)$, thus obtaining a representation for the net of ruled surfaces in planar coördinates.

For purposes of abbreviation let us write

$$t = |y, z_u, y_u|, \qquad t' = |z, z_u, y_u|. \tag{163}$$

We find from (162a) and (162b), by differentiation, making use of the equations of system (S), equations (6), and familiar theorems about determinants,

$$\begin{aligned} r_u &= -p_{11} r - p_{12} s + t, \\ r_v &= (b_{11} + b_{22} + c_{11}) r + c_{12} s - a_{12} t' + a_{22} t, \\ s_u &= -p_{21} r - p_{22} s + t', \\ s_v &= c_{21} r + (b_{11} + b_{22} + c_{22}) s + a_{11} t' - a_{21} t, \end{aligned} \tag{164}$$

whence follows, by elimination of t and t',

$$\begin{aligned} r_v = a_{22} r_u - a_{12} s_u &+ (b_{11} + b_{22} + c_{11} - a_{12} p_{21} + a_{22} p_{11}) r \\ &+ (c_{12} - a_{12} p_{22} + a_{22} p_{12}) s, \\ s_v = -a_{21} r_u + a_{11} s_u &+ (c_{21} + a_{11} p_{21} - a_{21} p_{11}) r \\ &+ (b_{11} + b_{22} + c_{22} + a_{11} p_{22} - a_{21} p_{12}) s. \end{aligned} \tag{165}$$

Let us put

$$\mathfrak{y} = rD^{-1/2}, \qquad \mathfrak{z} = sD^{-1/2} \tag{166}$$

where D is the fourth-order determinant

$$D = |y_u, z_u, y, z|, \tag{167}$$

whose partial derivatives are given in (157). The resulting system of differential equations may be written

$$
(168)\qquad
\begin{aligned}
\mathfrak{y}_{uu} + \mathfrak{p}_{11}\mathfrak{y}_u + \mathfrak{p}_{12}\mathfrak{z}_u + \mathfrak{q}_{11}\mathfrak{y} + \mathfrak{q}_{12}\mathfrak{z} &= 0,\\
\mathfrak{z}_{uu} + \mathfrak{p}_{21}\mathfrak{y}_u + \mathfrak{p}_{22}\mathfrak{z}_u + \mathfrak{q}_{21}\mathfrak{y} + \mathfrak{q}_{22}\mathfrak{z} &= 0,\\
\mathfrak{y}_v = \mathfrak{a}_{11}\mathfrak{y}_u + \mathfrak{a}_{12}\mathfrak{z}_u + \mathfrak{b}_{11}\mathfrak{y} + \mathfrak{b}_{12}\mathfrak{z}&,\\
\mathfrak{z}_v = \mathfrak{a}_{21}\mathfrak{y}_u + \mathfrak{a}_{22}\mathfrak{z}_u + \mathfrak{b}_{21}\mathfrak{y} + \mathfrak{b}_{22}\mathfrak{z}&.
\end{aligned}
$$

The values of the coefficients $\mathfrak{p}_{ik}$ and $\mathfrak{q}_{ik}$ may be transcribed from the theory of ruled surfaces.*

The values of $\mathfrak{a}_{ik}$ and $\mathfrak{b}_{ik}$ are easily computed from (165) and (166). We find

$$
(169)\qquad
\begin{aligned}
&\mathfrak{a}_{11} = a_{22}, \qquad \mathfrak{a}_{12} = -a_{12} \qquad \mathfrak{a}_{21} = -a_{21}, \qquad \mathfrak{a}_{22} = a_{11},\\
&\mathfrak{b}_{11} = b_{11} + \tfrac{1}{2}(a_{11} - a_{22})_u - \tfrac{1}{2}(a_{11} - a_{22})p_{11} + \tfrac{1}{2}a_{21}p_{12} - a_{12}p_{21},\\
&\mathfrak{b}_{12} = b_{12} + (a_{12})_u - (a_{11} - a_{22})p_{12} + \tfrac{1}{2}a_{12}(p_{11} - p_{22}) - a_{12}p_{22},\\
&\mathfrak{b}_{21} = b_{21} + (a_{21})_u + (a_{11} - a_{22})p_{21} - \tfrac{1}{2}a_{21}(p_{11} - p_{22}) - a_{21}p_{11},\\
&\mathfrak{b}_{22} = b_{22} - \tfrac{1}{2}(a_{11} - a_{22})_u + \tfrac{1}{2}(a_{11} - a_{22})p_{22} + \tfrac{1}{2}a_{12}p_{21} - a_{21}p_{12},\\
&\mathfrak{p}_{ik} = p_{ik},\\
&\mathfrak{q}_{11} = q_{11} + \tfrac{1}{4}(u_{11} - u_{22}), \qquad \mathfrak{q}_{12} = q_{12} + \tfrac{1}{2}u_{12},\\
&\mathfrak{q}_{21} = q_{21} + \tfrac{1}{2}u_{21}, \qquad \mathfrak{q}_{22} = q_{22} - \tfrac{1}{4}(u_{11} - u_{22}).
\end{aligned}
$$

It is an easy matter to verify the complete reciprocity between systems (S) and (168). We shall speak of them as systems *adjoint* to each other. In our interpretation the two systems correspond to the same congruence but in dualized representation. Clearly, if we re-interpret $\mathfrak{y}_1 \cdots \mathfrak{y}_4$ and $\mathfrak{z}_1, \cdots, \mathfrak{z}_4$ as point-coördinates, we obtain instead two nets of ruled surfaces which are dual to each other. Consequently those properties of a net whose analytic expressions remain unaltered by the transformations (169) are dualistic properties.

13. Conjugate systems of ruled surfaces

Let us put

$$
(170)\qquad \rho^{(u)} = 2y_u + p_{11}y + p_{12}z, \qquad \sigma^{(u)} = 2z_u + p_{21}y + p_{22}z.
$$

These expressions determine two points, $\rho^{(u)}$ and $\sigma^{(u)}$, such that the line joining them is a generator, of the same set as $P_y P_z$, on the quadric surface H which osculates the ruled surface $v = \text{const.}$ along $P_y P_z$. The lines which

* *Proj. Diff. Geom.*, p. 137.

join y to $\rho^{(u)}$, and z to $\sigma^{(u)}$ are generators of H of the second kind. Referred to the tetrahedron of these four points, and an appropriately chosen unit-point, the equation of H is

$$x_1 x_4 - x_2 x_3 = 0.* \tag{171}$$

We introduce two other points, by means of the expressions,

$$\rho^{(v)} = 2y_v + r_{11} y + r_{12} z, \qquad \sigma^{(v)} = 2z_v + r_{21} y + r_{22} z, \tag{172}$$

where the coefficients r_{ik} are defined by (15). These points are on the quadric H', which osculates the ruled surface $u =$ const. along $P_y P_z$, and their position on this quadric is determined by P_y and P_z by the same construction which, executed on H, gives rise to $\rho^{(u)}$ and $\sigma^{(u)}$.

Let us study the quadric H'. An arbitrary point on H' is given by an expression of the form

$$\beta_1 (\alpha_1 y + \alpha_2 z) + \beta_2 (\alpha_1 \rho^{(v)} + \alpha_2 \sigma^{(v)}). \tag{173}$$

But we have

$$\rho^{(v)} = 2(a_{11} y_u + a_{12} z_u + b_{11} y + b_{12} z) + r_{11} y + r_{12} z,$$

$$\sigma^{(v)} = 2(a_{21} y_u + a_{22} z_u + b_{21} y + b_{22} z) + r_{21} y + r_{22} z,$$

and

$$2y_u = \rho^{(u)} - p_{11} y - p_{12} z, \qquad 2z_u = \sigma^{(u)} - p_{21} y - p_{22} z,$$

whence

$$\rho^{(v)} = a_{11} \rho^{(u)} + a_{12} \sigma^{(u)} + (2b_{11} + r_{11} - a_{11} p_{11} - a_{12} p_{21}) y + (2b_{12} + r_{12} - a_{11} p_{12} - a_{12} p_{22}) z,$$

$$\sigma^{(v)} = a_{21} \rho^{(u)} + a_{22} \sigma^{(u)} + (2b_{21} + r_{21} - a_{21} p_{11} - a_{22} p_{21}) y + (2b_{22} + r_{22} - a_{21} p_{12} - a_{22} p_{22}) z,$$

or, on account of (37),

$$\begin{aligned} \rho^{(v)} &= a_{11} \rho^{(u)} + a_{12} \sigma^{(u)} + (r_{11} - \pi_{11}) y + (r_{12} - \pi_{12}) z, \\ \sigma^{(v)} &= a_{21} \rho^{(u)} + a_{22} \sigma^{(u)} + (r_{21} - \pi_{21}) y + (r_{22} - \pi_{22}) z. \end{aligned} \tag{174}$$

If we introduce these expressions into (173) we obtain an expression of the form

$$x_1 y + x_2 z + x_3 \rho^{(u)} + x_4 \sigma^{(u)},$$

where

$$\begin{aligned} x_1 &= \beta_1 \alpha_1 + \beta_2 \alpha_1 (r_{11} - \pi_{11}) + \beta_2 \alpha_2 (r_{21} - \pi_{21}), \\ x_2 &= \beta_1 \alpha_2 + \beta_2 \alpha_1 (r_{12} - \pi_{12}) + \beta_2 \alpha_2 (r_{22} - \pi_{22}), \\ x_3 &= \beta_2 \alpha_1 a_{11} + \beta_2 \alpha_2 a_{21}, \\ x_4 &= \beta_2 \alpha_1 a_{12} + \beta_2 \alpha_2 a_{22}, \end{aligned} \tag{175}$$

* *Proj. Diff. Geom.*, p. 191.

and where we may regard x_1, x_2, x_3, x_4 as the homogeneous coördinates of an arbitrary point of H' referred to the tetrahedron of the four points y, z, $\rho^{(u)}$, $\sigma^{(u)}$. To obtain the equation of H' we eliminate $\alpha_1, \beta_1, \alpha_2, \beta_2$ from (175). From the last two equations we find

$$a_2 \beta_2 \alpha_1 = a_{22} x_3 - a_{21} x_4, \qquad a_2 \beta_2 \alpha_2 = -a_{12} x_3 + a_{11} x_4$$

where we have put $a_2 = a_{11} a_{22} - a_{12} a_{21}$ as before. Substitution of these values into the first two equations of (175) gives

$$a_2 \alpha_1 \beta_1 = a_2 x_1 - (r_{11} - \pi_{11})(a_{22} x_3 - a_{21} x_4) - (r_{21} - \pi_{21})(-a_{12} x_3 + a_{11} x_4),$$

$$a_2 \beta_1 \alpha_2 = a_2 x_2 - (r_{12} - \pi_{12})(a_{22} x_3 - a_{21} x_4) - (r_{22} - \pi_{22})(-a_{12} x_3 + a_{11} x_4),$$

whence finally

$$\begin{aligned} &[a_2 x_1 + (\pi_{11} - r_{11})(a_{22} x_3 - a_{21} x_4) \\ &\qquad + (\pi_{21} - r_{21})(-a_{12} x_3 + a_{11} x_4)](-a_{12} x_3 + a_{11} x_4) \\ &- [a_2 x_2 + (\pi_{12} - r_{12})(a_{22} x_3 - a_{21} x_4) \\ &\qquad + (\pi_{22} - r_{22})(-a_{12} x_3 + a_{11} x_4)](a_{22} x_3 - a_{21} x_4) = 0 \end{aligned} \tag{176}$$

the equation of H'.

Let us consider any point P' of the line $P_y P_z$. Let the coördinates of such a point be x_1', x_2', 0, 0. The plane tangent to H' at P' will have the equation

$$-(a_{12} x_1' + a_{22} x_2') x_3 + (a_{11} x_1' + a_{21} x_2') x_4 = 0, \tag{177}$$

if we assume $a_2 \neq 0$. The corresponding tangent plane of H is given by

$$-x_2' x_3 + x_1' x_4 = 0. \tag{178}$$

These two planes will coincide, if and only if

$$a_{12} (x_1')^2 - (a_{11} - a_{22}) x_1' x_2' - a_{21} (x_2')^2 = 0,$$

that is, if P' is one of the two foci of the line $P_y P_z$.

Thus the two quadrics H and H' are tangent to each other at the foci. The two planes of contact are given by

$$a_{12} x_3^2 - (a_{11} - a_{22}) x_3 x_4 - a_{21} x_4^2 = 0. \tag{179}$$

Since all of the ruled surfaces of the net must be tangent to the focal sheets, it is clear that these two planes are the planes tangent to the focal sheets of the congruence, and we shall call them the focal planes of the line $P_y P_z$.

Let us write equation (178) in the form

$$x_3 - \lambda x_4 = 0,$$

where $\lambda = x_1' : x_2'$. This plane and the plane

$$x_3 - \mu x_4 = 0$$

will form a pair, given by

$$x_3^2 - (\lambda + \mu) x_3 x_4 + \lambda\mu x_4^2 = 0$$

which is divided harmonically by the focal planes (179), if and only if

(180) $$2a_{12}\lambda\mu - (a_{11} - a_{22})(\lambda + \mu) - 2a_{21} = 0,$$

whence

$$\mu = \frac{2a_{21} + (a_{11} - a_{22})\lambda}{-(a_{11} - a_{22}) + 2a_{12}\lambda}.$$

The plane (177) will coincide with the plane $x_3 - \mu x_4 = 0$ for all values of λ, if and only if the equation

$$[(a_{11} - a_{22})\lambda + 2a_{21}](a_{12}\lambda + a_{22}) = [2a_{12}\lambda - (a_{11} - a_{22})](a_{11}\lambda + a_{21})$$

is an identity, that is, if

$$a_{11}^2 = a_{22}^2, \qquad (a_{11} - a_{22})a_{12} = 2a_{11}a_{12}, \qquad 2a_{21}a_{22} = -a_{21}(a_{11} - a_{22}).$$

If a_{12} and a_{21} are not both equal to zero, we conclude

$$a_{11} + a_{22} = 0.$$

If a_{12} and a_{21} are both zero, these conditions are satisfied also by

$$a_{12} = a_{21} = a_{11} - a_{22} = 0;$$

but in this case the congruence degenerates into a single ruled surface.

Let us agree to speak of a net of ruled surfaces as a *conjugate net* if the osculating quadrics, H and H', of the two ruled surfaces $u =$ const. and $v =$ const. which meet along a line $P_y P_z$ of the congruence, are so related that the planes tangent to H and H' respectively at every point P of $P_y P_z$ are divided harmonically by the focal planes of $P_y P_z$. We have just shown that *the net of ruled surfaces defined by system* (S) *is a conjugate net, if and only if*

(181) $$a_1 = a_{11} + a_{22} = 0,$$

provided that (S) defines a proper net of ruled surfaces which does not degenerate into a single ruled surface.

Thus the method of Article 8 for the construction of invariants of a one-parameter family of ruled surfaces consists geometrically of associating with the given one-parameter family of ruled surfaces its conjugate family; the invariants of the conjugate net obtained in this way are at the same time the invariants of the given one-parameter family.

Clearly the developables of the congruence are their own conjugates. Thus, making use of this new notion of conjugate systems of ruled surfaces, the analogy between the developables of a congruence and the asymptotic lines on a surface becomes very evident.

Since the quadrics H and H' have a line of the congruence in common, the rest of their intersection will be, in general, a twisted cubic. We shall refrain, at present, from any discussion of the congruence of cubics which arises in this way.

14. Geometric significance of some of the invariants obtained previously

The factors of the covariant

$$a_{21}\,y^2 - (a_{11} - a_{22})\,yz - a_{12}\,z^2 \tag{182}$$

determine the foci of the line $P_y\,P_z$. The factors of

$$u_{21}\,y^2 - (u_{11} - u_{22})\,yz - u_{12}\,z^2 \tag{183}$$

determine the flecnodes of $P_y\,P_z$ when $P_y\,P_z$ is regarded as a generator of a ruled surface v = const. The factors of

$$\begin{aligned}[(u_{11} - u_{22})\,v_{21} - (v_{11} - v_{22})\,u_{21}]\,y^2 + 2\,(u_{12}\,v_{21} - u_{21}\,v_{12})\,yz \\ + [(u_{11} - u_{22})\,v_{12} - (v_{11} - v_{22})\,u_{12}]\,z^2\end{aligned} \tag{184}$$

determine the complex points of $P_y\,P_z$, if again $P_y\,P_z$ be regarded as a generator of a ruled surface of the family v = const.*

The discriminants of these three quadratic covariants are A, $\theta_{4,\,0}$, and $4\theta_{10,\,0}$, and the significance of the vanishing of these invariants is therefore apparent.

The bilinear invariant of (182) and (183) is

$$-\tfrac{1}{2}(a_{11} - a_{22})(u_{11} - u_{22}) - a_{12}\,u_{21} - a_{21}\,u_{12} = \theta_{1,\,1},$$

according to (59). Therefore, *the flecnodes of a generator of a ruled surface* v = *const., and the foci of that line separate each other harmonically, if and only if* $\theta_{1,\,1} = 0$.

The bilinear invariant of (182) and (184) is

$$\begin{aligned}a_{21}[(u_{11} - u_{22})\,v_{12} - (v_{11} - v_{22})\,u_{12}] - a_{12}[(u_{11} - u_{22})\,v_{21} - (v_{11} - v_{22})\,u_{21}] \\ + (a_{11} - a_{22})(u_{12}\,v_{21} - u_{21}\,v_{12}),\end{aligned}$$

and this is equal to

* *Proj. Diff. Geom.*, p. 208.

$$\begin{vmatrix} a_{11} - a_{22}, & a_{12}, & a_{21} \\ u_{11} - u_{22}, & u_{12}, & u_{21} \\ v_{11} - v_{22}, & v_{12}, & v_{21} \end{vmatrix} = (a, u, v) = \theta_{4,1}$$

according to (60).

Consequently *the complex points of a generator of a ruled surface $v = const.$, and the foci of the same line separate each other harmonically if and only if* $\theta_{4,1} = 0$.

The bilinear invariant of (182) and (184) vanishes identically, corresponding to the fact that the flecnodes and complex points always separate each other harmonically. The quantity

$$\frac{\theta_{1,1}^2}{A\theta_{4,0}} \tag{185}$$

is an absolute invariant of the one-parameter family of ruled surfaces which determines the cross ratio α of the flecnodes and foci by means of the equation

$$\theta_{1,1}^2(\alpha - 1)^2 - A\theta_{4,0}(\alpha + 1)^2 = 0. \tag{186}$$

The form of (184) suggests a new covariant

$$\begin{aligned} [(u_{11} - u_{22})a_{21} - (a_{11} - a_{22})u_{21}]y^2 + 2(u_{12}a_{21} - u_{21}a_{12})yz \\ + [(u_{11} - u_{22})a_{12} - (a_{11} - a_{22})u_{12}]z^2. \end{aligned} \tag{187}$$

Clearly the bilinear invariant of (182) and (187) is $(a, u, a) = 0$, and that of (183) and (187) is $(a, u, u) = 0$. Consequently (187) represents the pair of points on $P_y P_z$ which are harmonic conjugates of each other both with respect to the flecnodes and the foci of $P_y P_z$. The discriminant of (187) is a new invariant of the one-parameter family of ruled surfaces.

15. The Laplace transformation

The first Laplace transformation is given by

$$\eta_1 = \eta_{\bar{u}} - \frac{m_{\bar{u}}}{m}\eta = m\frac{\partial}{\partial \bar{u}}\left(\frac{\eta}{m}\right), \qquad \zeta_1 = \frac{\eta}{m}. \tag{188}$$

We wish to express these covariants in terms of the variables and coefficients of system (S). We find, from (188) and (136),

$$\eta_1 = -\frac{\omega'\beta_{12}}{\omega\psi_u}\frac{\partial}{\partial\bar{u}}\left(\frac{-\bar{y}\psi_u}{\omega'\beta_{12}}\right) = \frac{\omega'\beta_{12}}{\omega\phi_u\psi_u}U\left(\frac{\psi_u\bar{y}}{\omega'\beta_{12}}\right).$$

But we have found

$$U(\psi_u) = \alpha_{22}\psi_u, \qquad U(\omega') = \beta_{22}\omega',$$

so that we find

$$\omega\phi_u\eta_1 = \beta_{12}U\left(\frac{\bar{y}}{\beta_{12}}\right) + (\alpha_{22} - \beta_{22})\bar{y}, \tag{189}$$

where $\bar{y}$ is given by (98), and

$$\zeta_1 = -\frac{\psi_u \bar{y}}{\omega' \beta_{12}}. \tag{190}$$

In the same way we find for the minus first Laplace transformation

$$\eta_{-1} = \frac{\phi_u \bar{z}}{\omega\beta_{21}}, \qquad \omega' \psi_u \zeta_{-1} = \beta_{21} V\left(\frac{\bar{z}}{\beta_{21}}\right) - (\alpha_{11} - \beta_{11})\bar{z}, \tag{191}$$

where the value of $\bar{z}$ is also given by (98).

The invariants h_{-1} and k_{-1} have been calculated before. (See equations (147) to (152)), and we have

$$h_1 = k = mn, \qquad k_1 = m_1 n_1 - \frac{\partial^2 \log m_1}{\partial \bar{u} \partial \bar{v}} = k - \frac{\partial^2 \log km}{\partial \bar{u} \partial \bar{v}}, \tag{192}$$

where, on account of (149),

$$2\phi_u \psi_u \frac{\partial^2 \log km}{\partial \bar{u} \partial \bar{v}} = UV(km) + VU(km) + \alpha_{11} U(km) - \alpha_{22} V(km). \tag{193}$$

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